Discrete Invasive Weed Optimization for Nash Equilibrium Search in Nonlinear Games

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Abstract— **Finding Nash Equilibrium (NE) for nonlinear games is a challenging work due to existence of local Nash equilibrium traps. So, devising algorithms that are capable of escaping from local optima and finding global solutions is needed for analysis of nonlinear games. Evolutionary algorithms (EAs) as the popular stochastic global search algorithms can be exploited for this purpose. This paper studies performance of discrete invasive weed optimization (DIWO) for Nash equilibrium search in games with numerous local NEs. Firstly, DIWO is introduced and specialized for optimization in descretized spaces. Efficiency of DIWO for function optimization is evaluated and compared with some other discrete EAs through a number of popular test functions in stochastic optimization literature. Afterwards, it is explained how to characterize NEs as minima of an objective function, and DIWO is used to minimize these functions in two experimental studies. The first problem is a static nonlinear game with multiple local NEs, and the second one is Cournot model of a transmission-constrained electricity market called IEEE30 bus test system. The results show that the proposed algorithm is a promising technique to come up with complex theoretical and practical games.**

Index Terms—**Evolutionary Algorithms, Discrete Invasive Weed Optimization, Nash Equilibrium, Electricity Markets.**

I. INTRODUCTION

MANY techniques have been developed for searching Nash Equilibrium (NE) in game theory problems. All the approaches are inspired by NE definition which is maximizing the payoff, given other players' strategies. The simplest method which can be applied to two or three player games is finding the intersection of best response curves (reaction curves) by drawing or Algebra. For graphical approach, some geometric techniques have been also proposed to come up with more than two player problems [1]. Algebra can improve the method to solve games with several players, but it can be applied to problems with simple mathematical manipulations. This algorithm is commonly used in Cournot or Bertrand models of electricity markets with linear demand functions, using the first-order condition for maximizing each player's payoff [2], [3], [4].

Iterative NE search in which players repeatedly maximize their payoff by turn is another method that is applied to more complex problems. The profit maximization problem which is embedded in this method can be solved by local or global optimization algorithms. In literature, local search is more popular and have been employed in [5], [6] and [7], however in [8], a GA-based algorithm is also presented for profit maximization.

In recent years, with development of Soft Computing, and increasing growth of bioinspired computing in a variety of applications, a considerable amount of attention has been dedicated to evolutionary programming and computational intelligence for game learning and simulation of games in electricity markets [5], [8]-[16]. Coevolutionary programming is the most popular technique for this purpose. In [5], a novel Hybrid Coevolutionary is applied to solve constrained-transmission electricity markets, and in [11], a GA-based coevolutionary algorithm is exploited to simulate a simple electricity pool. Besides coevolutionary algorithms, learning methods in agent-based approach have been employed to study imperfect competition in electricity markets [17]-[19]. In fact, these days, agent-based economics is a rigorous opponent of game theory to simulate electricity markets.

Another approach for searching NE is characterization of NEs in terms of minima of a function and then minimizing this objective function. This method was firstly employed in finding mixed strategy NEs [12], [13], but recently a similar technique has been introduced in [10] to identify pure NE in games with a large number of players. The virtue of this approach is that it provides a measure to evaluate fitness of an obtained NE and also prepares a basin to find all NEs for a game with more than one NE using the conventional techniques posed in optimization literature. It seems that more investigations are needed to understand the efficiency of this approach (which is partially addressed in this paper).

In this study, Invasive Weed Optimization (IWO) algorithm as an efficient evolutionary algorithm for fast and global search is employed to find NE in complex nonlinear games. In fact, discrete invasive weed optimization (DIWO) which was proposed in the previous work for combinatorial optimization [20] is modified and specialized for search in descretized spaces and used to minimize objective functions which encode NEs as their minima. Invasive Weed Optimization is a novel ecologically inspired algorithm that mimics the process of weeds colonization and distribution. Despite its recent development, it has shown successful results in a number of practical applications like optimization and tuning of a robust

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controller [21], optimal positioning of piezoelectric actuators [22], developing a recommender system [23], distributed identification and adaptive control of a surge tank [24], analysis of electricity markets dynamics [25], cooperative task assignment of UAVs [20], Nash equilibrium search in electricity markets [26], etc.

Section II provides steps for algorithm design comprising quick review of continuous IWO, introduction to discrete IWO, and discussion on parameters setting for DIWO. In section III, simulation results for optimization of some famous benchmarks are presented and compared with two other discrete EAs. Section IV is dedicated to Nash equilibrium search with minimization of an objective function. This section starts by a summary of some basics like description of games and Nash equilibrium, and next, it is explained how to define the objective function. In addition, two experiments are conducted in this section including a nonlinear game with multiple local minima and a transmission-constrained electricity market model known as IEEE30 bus test system to evaluate performance of the proposed algorithm. Finally, conclusions are drawn and future works are summarized in section V.

II. Algorithm Design

A. Continuous Invasive Weed Optimization

IWO was developed by Mehrabian and Lucas in 2006 [21]. IWO algorithm is a numerical stochastic search algorithm mimicking natural behavior of weeds in colonizing and finding suitable place for growth and reproduction. Some of distinctive properties of IWO in comparison with other EAs are the way of reproduction, spatial dispersal and competitive exclusion [21].

In IWO, the process begins with initializing a population. It means that a population of initial solutions is randomly generated over the problem space. Then each member of population produces seeds depending on its relative fitness in the population. Number of seeds for each member varies between S_{min}, for the worst member of population, and S_{max}, for the best member of population. Seeds are randomly scattered in solution space by normally distributed random numbers with mean equal to zero. Standard deviation (SD) of normal distribution for each generation is determined by (1).

$$\sigma_{iter} = \frac{(iter_{max} - iter)^n}{(iter_{max})^n}\left(\sigma_{init} - \sigma_{final}\right) + \sigma_{final} \qquad (1)$$

$iter_{max}$ is the maximum number of iterations, σ_{iter} is the SD at the current iteration and n is the nonlinearity modeling index. The produced seeds and their parents considered as the potential solutions for the next generation. Finally, after a number of iterations the population reaches its maximum and an elimination mechanism should be employed. For this purpose, the seeds and their parents ranked together and those with better fitness survive and become reproductive [21]. The pseudocode for IWO is presented in Fig. 1, and the set of parameters for IWO algorithm is provided in Table I.

1. Genearte random population of N_0 solutions;
2. For $iter$=: 1 to the maximum number of generations
   a. Compute maximum and minimum fitness in the colony;
   b. For each individula $w \in W$
      i. Compute number of seeds for w according to its fitness;
      ii. Randomly distribute generated seeds over the search space with normal distribution around the parent plant w;
      iii. Add the generated seeds to the solution set, W;
   c. If $(|W| = N) > p_{max}$
      i. Sort the population W in descending order of their fitness;
      ii. Truncate population of weeds with smaller fitness until $N = P_{max}$;
3. Next $iter$;

Figure 1. Pseudocode for IWO algorithm

TABLE I. IWO Parameters

Symbol	Definition
N_0	Number of initial population
$iter_{max}$	Maximum number of iterations
P_{max}	Maximum number of plants
S_{max}	Maximum number of seeds
S_{min}	Minimum number of seeds
n	Nonlinear modulation index
σ_{init}	Initial vale of standard deviation
σ_{final}	Final vale of standard deviation

B. Discrete Invasive Weed Optimization

Due to continuous IWO's distinctive properties, its local and global abilities for exploration and exploitation, and also its successful results in a considerable number of applications after a short time of its development, Discrete Invasive Weed Optimization (DIWO) was proposed in [20] hopping to exploit these features in discrete optimization problems. The algorithm introduced in [20] was applied to combinatorial optimization problems and was somehow heuristic in some aspects. However, here, we provide a scheme which has a clear and straightforward procedure for optimization in descretized spaces.

The framework for DIWO is the same as IWO's, but some considerations are taken for exploration in discrete search spaces. The psuedocode for DIWO is given in Fig. 2.

1. Genearte random population of N_0 plants from the set of pheasible solutions;
2. For $iter$=: 1 to the maximum number of generations
   a. Compute maximum and minimum fitness in the colony;
   b. For each plant $w \in W$
      i. Compter number of seeds of w, corresponding to its fitness
      ii. Randomly select the seeds from the pheasible solutions around the parent plant (w) in a neighborhood of radius R with normal distribution
      iii. Add the generated seeds to the solution set, W
   c. If $(|W| = N) > p_{max}$
      i. Sort the population N in descending order of their fitness
      ii. Truncate population of weeds with smaller fitness until $N = p_{max}$
3. Next $iter$;

Figure 2. Psuedocode for DIWO algorithm

The process for computing number of seeds and also competition exclusion is completely the same as IWO, but seeds generation has been modified to random selection of solutions from the hypercube of radius R in the dim-dimentional space of feasible solutions around the plant with a normal distribution. For purpose of optimization in evenly descretized spaces with grids (which is the case, here), neighborhood is defined with cells in the grid world. It means that for a reproductive plant, all the neighboring cells within the hypercube of size $(2R-1)^{dim}$ around the plant are considered and each dimension of seed is generated by randomly selecting a cell from $2R-1$ potential cells with weights normally decaying from origin. This process is illustrated in Fig. 3.

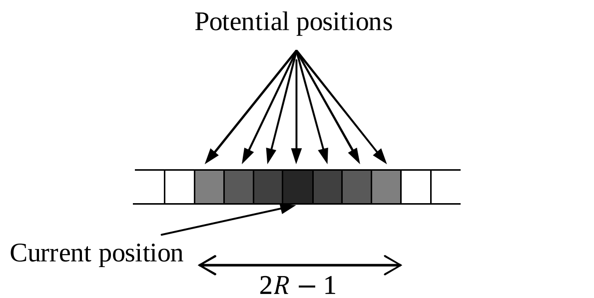


Figure 3. A 1-D descretized space and random weitghted selection

For parameters setting, following the guidelines presented in [21] and [24] and also our experimental studies, some suggestions can be offered. Firstly, the best and general value for n is 3. It is suggested to set S_{min} to 0 or 1 and S_{max} to 3. R, σ_{init}, and σ_{final} are fixed according to the problem range of solutions. We suggest to set R for each dimension to $1/5$ of the total cells in that dimension after dividing the initial solution space into grids.

III. Simulation Results of DIWO for Function Optimization

A. Convergence of DIWO

Three studies are conducted to demonstrate evolutionary process of optimization in DIWO to locate global minima of discrete functions. The benchmarks are Sphere, Griewank, and Rastrigin functions which are described in Table II.

TABLE II. Benchmark Functions

Name	Function	Limits	n	Prec.
Sphere	$f(x) = \sum_{i=1}^{n}(x_i^2)$	[-40, 40]	2	0.01
Griewank	$f(x) = 1 + \sum_{i=1}^{n}\frac{x_i^2}{4000} - \prod_{i=1}^{n}(\cos(x_i/\sqrt{i}))$	[-5.12, 5.11]	10	0.01
Rastrigin	$f(x) = \sum_{i=1}^{n}(x_i^2 - 10\cos(2\pi x_i) + 10)$	[-5.12, 5.11]	30	0.01

The Sphere function is quadratic, continuous, convex, and unimodal. The minima for this function is 0 at the origin. This function provides an easily analyzable first test for the optimization algorithm. Process of colonization of weeds around the point with the best fitness is shown in Fig. 4. It can be observed that the plants grow towards the optimal point from the initialization area. In their progress towards the optimal point, plants with worse fitness are being excluded, and only weeds with better fitness are allowed to be reproduced, which leads in colonization about the optimal point.

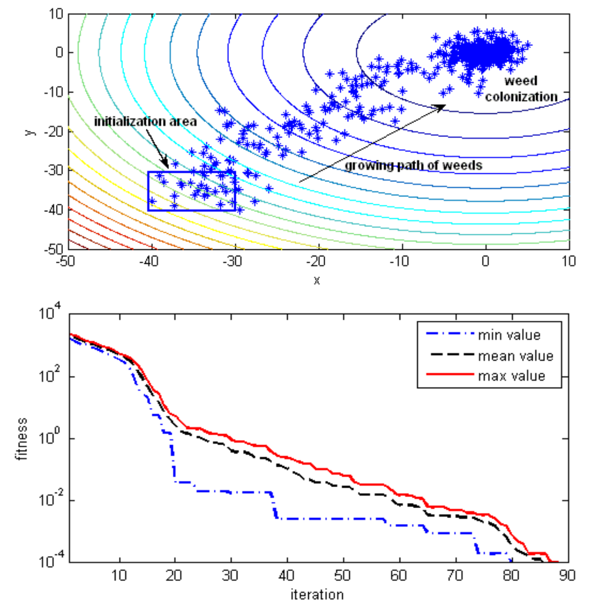


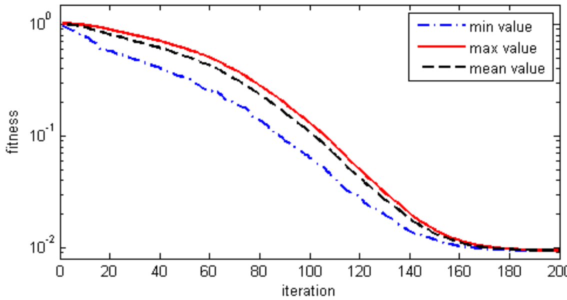
Fig. 4. Convergence of DIWO to the optimal value of the Sphere function

TABLE III. DIWO Parameters for Sphere Function Minimization

Symbol	value	Symbol	value
N_0	10	n	3
$iter_{max}$	100	σ_{init}	500
P_{max}	10	σ_{final}	1
S_{max}	3	R	500
S_{min}	0	X_{init}	[-40, -30]

The second benchmark is Griewank function with dimension of 10. This is a multimodal function with a global minimum at origin which is commonly used to evaluate performance of EAs for global optimization. To show virtues of DIWO for stochastic optimization, we compare the proposed algorithm with discrete particle swarm optimization (DPSO) [27] and genetic algorithm (GA) with binary encoding. All the algorithms are conducted for thirty times with approximately the same number of function evaluations to have a fair comparison. Average results of the experiments are depicted in Fig. 5, showing superiority of DIWO for optimization of this function.

Finally, the third benchmark is Rastrigin function which is non-convex, and multimodal. It is a fairly difficult problem for evolutionary algorithms due to the large number of local minima. Like previous functions, the global minimum is located at origin. Again, DIWO is compared with DPSO and GA (binary) in this study. Each algorithm run is repeated for thirty times with nearly the same number of function evaluations for the purpose of fair comparison. Average results of this experiment are presented in Fig. 6. It can be observed that DIWO outperforms GA, but DPSO has a little better fitness than DIWO for optimization of this benchmark.



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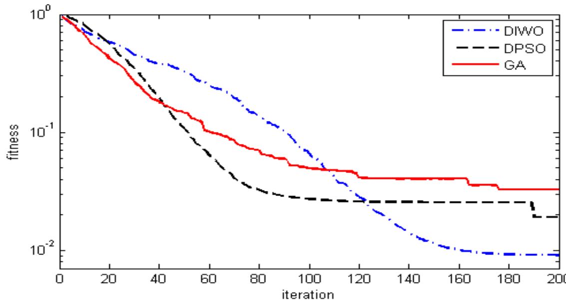
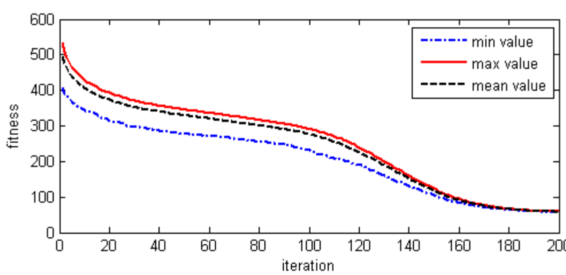


(b)

Fig. 5. Optimization process of Griewank function
a) Evolution of fitness function
b) Comparison of DIWO with DPSO and GA



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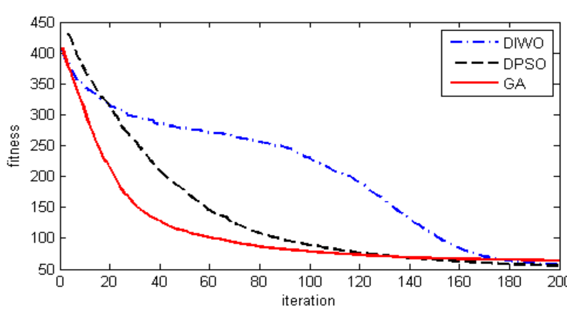


(b)

Fig. 6. Optimization process of Rastrigin function
a) Evolution of fitness function
b) Comparison of DIWO with DPSO and GA

TABLE IV. DIWO PARAMETERS FOR GRIEWANK AND RASTRIGIN FUNCTION MINIMIZATION

Symbol	value		Symbol	value	
	Griewank	Rastrigin		Griewank	Rastrigin
N_0	22	40	n	3	3
$iter_{max}$	200	200	σ_{init}	250	200
P_{max}	22	40	σ_{final}	1	1
S_{max}	3	3	R	250	200
S_{min}	1	1	X_{init}	[-5.12, 5.11]	[-5.12, 5.11]

B. Experimental Studies on De Jong's test suite

In this part, we employ the very standard De Jong's Test Suite in [28] which provides functions with different archetypes to evaluate performance of DIWO. These functions are characterized in Table V. The first function is Sphere function introduced in the previous part. The Rosenbrock function (F2) is quadratic, continuous, non-convex, unimodal. For n dimensions, this function has its global minima at $\mathbf{1}_{n\times 1}$. This function is considered as a nightmare for most of the optimization algorithms because it has deep parabolic valley along the curve. Algorithms that are not able to discover good directions underperform in this problem. The third function is Step function which is discontinuous, non-convex, and unimodal. This function is used as a representative of problems with flat surfaces that are considered as obstacles for optimization algorithms, because they do not give any information about which is the feasible direction. The main idea of this function is to make the search more difficult by introducing small plateaus to the topology. The Quartic function (F4) is quadratic, continuous, convex, unimodal padded with Gaussian noise. The fact of introducing noise to the function causes that the algorithm never gets the same value on the same point. Algorithms that do not work well optimizing this function will work poorly on surfaces with noisy data. Note that in this study, fitness values for this function are presented with mean of objective values for the individuals in each generation (not the best). The last function is the Foxholes function which is continuous, non-convex, non-quadratic, two-dimensional with 25 local minima and value of approximately 1 for these points. The results are compared with those of DPSO and GA (binary) reported in [29]. It is tried to have the same number of function evaluations in different algorithms for the purpose of fair comparison. Results of experiments including best achieved values and standard deviations (in parenthesis) and the employed parameters are provided in Table VI and Table VII respectively. It can be observed that DIWO manages to obtain good final values and surpasses the performance of other algorithm in all the test functions except for Step function (F3).

TABLE V. DE JONG'S TEST SUITE

Fn.	Function	Limits	n	Precision
F1	$f(x) = \sum_{i=1}^{n}(x_i^2)$	[-5.12, 5.11]	3	0.01
F2	$f(x) = \sum_{i=1}^{n}[100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2]$	[-5.12, 5.11]	2	0.01
F3	$f(x) = 6n + \sum_{i=1}^{n}(\lfloor x_i \rfloor)$	[-5.12, 5.11]	5	0.01
F4	$f(x) = GAUSS(0,1) + \sum_{i=1}^{n}(i \,.\, x_i^4)$	[-1.28, 1.27]	30	0.01
F5	$\frac{1}{f(x)} = \frac{1}{500} + \sum_{j=0}^{24} \frac{1}{j+1+(x_1-a_{jmod5})^6+(x_2-a_{j/5})^6}$	[-65.5, 65.5]	2	0.001

TABLE VI. COMPARING DIWO WITH DPSO AND GA ON DE JONG'S TEST SUITE

Fn.	GA	DPSO	DIWO	EN[a] for DIWO	EN for others
F1	0.00014 (0.00009)	0.00008 (0.00007)	0.00000 (0.00000)	6714	8000
F2	0.27285 (0.41788)	0.10702 (0.17433)	0.00000 (0.00000)	7686	8000
F3	0.00000 (0.00000)	0.00000 (0.00000)	0.02000 (0.15683)	7998	8000
F4	7.13937 (2.19431)	2.52286 (1.08721)	0.94604 (0.46457)	5142	8000
F5	1.17155 (0.14406)	1.03724 (0.08504)	1.00794 (0.09940)	16024	16000

a. mean of function evaluation number

TABLE VII. DIWO PARAMETERS FOR DE JONG'S TEST SUITE OPTIMIZING

Symbol	value	Symbol	value
N_0	20	n	3
$iter_{max}$	200, 400[e]	σ_{init}	100[a,b], 300[c], 50[d], 40000[e]
P_{max}	20, 23[e]	σ_{final}	1
S_{max}	3	R	100[a,b], 200[c], 50[d], 40000[e]
S_{min}	1	-	-

a. For F1; b. For F2; c. For F3; d. For F4; e. For F5;

IV. DIWO FOR NE SEARCH

A. Games and Nash Equilibrium

A general multi-player game consists of an index set $N = \{1, 2, 3, \dots, N\}$ called player's set and an index set $K = \{1, 2, 3, \dots, K\}$ as the stages of the game, showing the allowable number of moves for each player. In each stage, players take strategies from a set of strategy spaces $U = \{U_k^i\}$, and receive a payoff of $\pi_i(u^i, u^{-i})$, where $u^i \in U^i$ is the pure strategy for player i, given pure strategy set of others $u^{-i} = \{u^1, \dots, u^{i-1}u^{i+1}, \dots, u^N\} \in U^{-i}$. Pure strategy Nash Equilibrium (NE) is a point where no player can obtain a higher profit by unilateral movement. The satisfying NE condition for the combined strategy $\{u^{i*}, u^{-i*}\}$ is characterized in (2).

$$\forall i, \forall u^i \in U^i, \qquad \pi_i(u^{i*}, u^{-i*}) \geq \pi_i(u^i, u^{-i*}) \tag{2}$$

As we will use the term local NE in this paper, here a definition of that from [5] is also provided.

$$\exists \varepsilon > 0 \text{ such that } \forall i, \forall u^i \in B^{i,\varepsilon}(u^{i*}), \quad \pi_i(u^{i*}, u^{-i*}) \geq \pi_i(u^i, u^{-i*}) \tag{3}$$

where $B^{i,\varepsilon}(\hat{u}^i) = \{u^i \epsilon U^i \ \left\| u^i - \hat{u}^i \right\| < \epsilon\}$

B. Nash Equilibrium as a Minimum of a Function

The idea of characterization of Nash equilibria in terms of minima of a function was developed in [10], although a similar approach was previously used in [12] and [13] for identifying mixed NEs in games. In this method, an objective function is defined in which the minima are the NEs, and then any optimization algorithm can be exploited to solve this minimization problem. However, this objective function is driven in an indirect process that makes hard for the local optimization algorithms to find the minima, so a stochastic optimization algorithm should be used.

One of the advantages of constructing an objective function is that we have a measure to assess merit of obtained solutions by its fitness value in the objective function. The other advantage is that we can apply conventional techniques like deflection, stretching, repulsion, etc. in optimization for computation of all NEs [12], [13], [30].

The objective function for each combined strategy u in the strategy space U and payoff function π is defined as follows:

$$J(u) = \sum_{i=1}^{N} [\max_{u_i' \in U_i} \pi_i(u_1, \dots, u_{i-1}, u_i', u_{i+1}, \dots, u_N) - \pi_i(u)] \tag{4}$$

Form the classical definition of Nash Equilibrium, it is easily concluded that the function J is strictly positive, if the combined strategy u is not equilibrium and equal to zero otherwise, so the NEs are the minima of this function.

It can be observed in (4) that a maximization problem is embedded in this function, for which direct exhaustive search, local or global optimization can be employed. In games with discrete and not too large strategy spaces, maximization can be performed by sorting the payoffs, but for continuous games, local or global maximization might be useful. In this study, we are involved with games where the strategy spaces are discretized to small grids with arbitrary precision. So, the objective function is easily calculated for each strategy by exhaustive search and then the proposed EA is used to minimize this function.

C. A Numerical Example

This is a nonlinear static game with *local NE traps* [5], which is also analyzed in [5] and [8], and we can consider it as a good benchmark for nonlinear games. The profit function for this game is characterized in (5), and the global best responses and the local best responses for this game are illustrated in Fig. 7.

$$\begin{aligned} \pi_1(x_1, x_2) &= 21 + x_1 \sin(\pi\, x_1) + x_1\, x_2 \sin(\pi\, x_2) \\ \pi_2(x_1, x_2) &= 21 + x_2 \sin(\pi\, x_2) + x_1\, x_2 \sin(\pi\, x_1) \end{aligned} \tag{5}$$

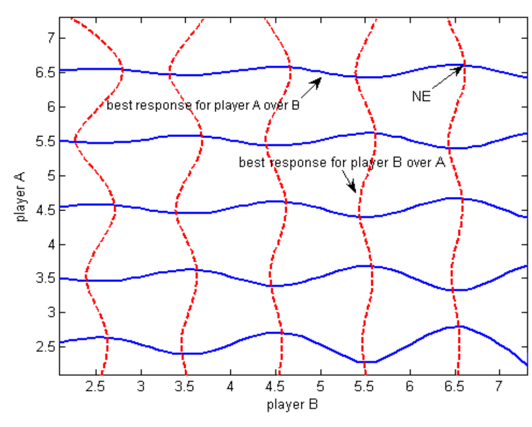
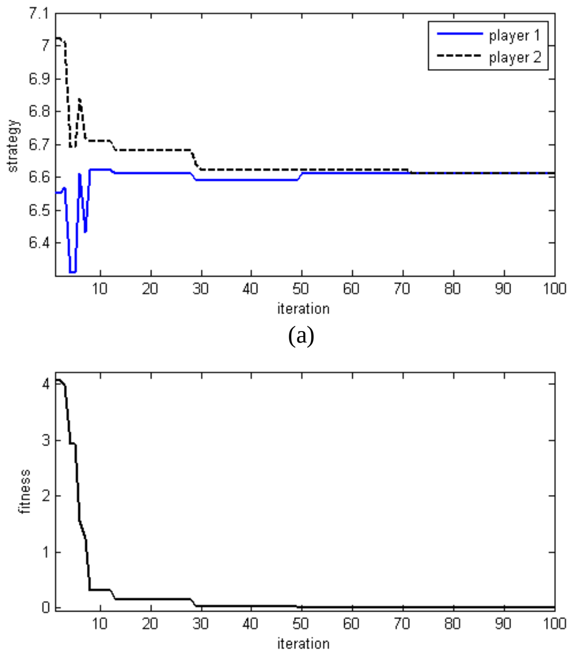


Fig. 7. Local and global best responses for the numerical example.

We evaluate performance of our proposed discrete invasive weed optimization (DIWO) for NE search in this nonlinear game with strategies descretized at precision of 0.1. Fig. 8a shows the strategies evolution while Fig. 8b presents trace of fitness values for the objective function through the evolutionary process. It can be seen that the algorithm is capable of identifying the global NE with the specified precision. Comparing with experiments in [5], we can say that our algorithm is better than the simple coevolutionary genetic algorithm in finding the global NE.



(a)

(b)

Fig. 8. NE search for a nonlinear static game with DIWO
a) Strategies evolution
b) Objective function minimization

TABLE VIII. DIWO PARAMETERS FOR NE SEARCH IN THE NUMERICAL EXAMPLE

Symbol	value	Symbol	value
N_0	20	n	3
$iter_{max}$	100	σ_{init}	150
P_{max}	20	σ_{final}	1
S_{max}	3	R	150
S_{min}	1	-	-

D. *Transmission-Constrained Electricity Markets*

Although transmission-constrained electricity markets with linear demand functions have linear demand curves, but the transmission constraints can cause individual profit functions to have local optima [7]. Actually, reaction curves in this model are discontinuous piecewise linear functions that might make local NE traps [5] or even disrupt existence of pure strategy equilibrium for the game [5], [31], [32]. Besides the fact that transmission-constrained electricity market model is a good mathematical example with a complex game structure and local optima, it is an important model for market power analysis in the restructured electricity industry [32], [33]. Hence, transmission-constrained electricity market is a good example of complex game for our purpose of Soft Computing. Shortly, trading in electricity markets can be represented by the maximization of total welfare subject to the constraints on the system (6).

$$\max\left(\sum_j Benefit_j - \sum_i Cost_i\right)$$

$$S.T.\ \begin{cases} Transmission\ thermal\ limits \\ Total\ supply = total\ demand \\ Kirchoff's\ laws \end{cases} \quad (6)$$

When transmission constraints are binding in the imperfectly competitive market, Cournot behavior will produce locational price differences similar to a competitive market with constraints present. This increases the difficulty of computing the profit maximizing condition of the strategic players. The profit maximizing function of each strategic player has an embedded transmission-constrained welfare maximization problem within its major problem. The generation and transmission line constraints are included in the welfare maximization subproblem. The profit function maximization of each utility is given in (7).

$$\max\left\{P_i\, q_i - Cost_i \;\middle|\; \begin{matrix} \max\sum_j Benefit_j\ , \\ Transmission\ Constraints \end{matrix} \right\} \quad (7)$$

Locational prices (P_i), are determined by the Lagrange multipliers of the locational energy balance equality condition for Kirchoff's laws in the welfare maximization problem which is also the market-clearing problem, here [34], [35].

E. *IEEE30 Bus System with Transmission Constraints*

To illustrate the performance of DIWO in a practical and complicated problem, IEEE30 bus test system is studied in this part. IEEE30 bus network (cf. Fig. 9) is composed of six producers and 20 consumers. We consider three transmission constraints for this network which are listed in Table IX. Data for producers and load demand curves are provided in Table X and XI respectively. Note that load demand curves for this system are linear functions modeled according to (8). With these settings, the system has one pure NE at $\boldsymbol{q}$ = (26.6, 45.4, 36.8, 24.2, 43.4, 27.9) with $\pi(\boldsymbol{q})$ = (447, 896, 731, 505, 1054, 800).

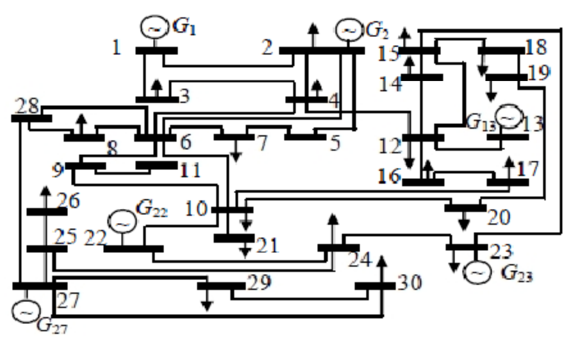


Fig. 9. IEEE30 bus test system [31]

$$p_i = e_i + f_i \times d_i \quad (8)$$

TABLE IX. LINES WITH TRANSMISSION CONSTRAINTS

Line	From bus	To bus	Flow limit (MW)
7	4	6	5
25	10	20	5
33	24	25	5

TABLE X. PRODUCERS' COST DATA

Cost Function	Bus	a_i	b_i	q_i^{min}	q_i^{max}
$C_i(q_i) = a_i q_i + \frac{1}{2} b_i q_i^2$	#1	25	0.15	5	80
	#2	20	0.25	5	60
	#13	23	0.2	5	60
	#22	22	0.25	5	60
	#23	20	0.2	5	80
	#27	22	0.15	5	70

TABLE XI. DATA FOR LOAD DEMAND FUNCTIONS

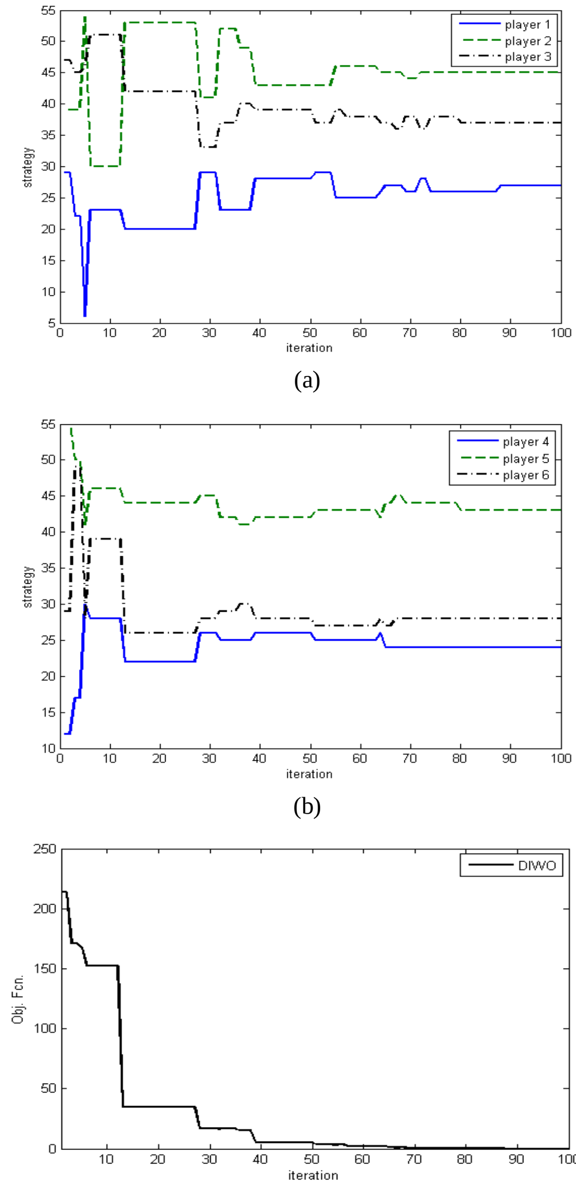
Bus	e_i \$/MW	f_i \$/MW^2	Bus	e_i \$/MW	f_i \$/MW^2
2	125	-5	17	100	-4.5
3	80	-4	18	80	-4
4	100	-4	19	100	-5
7	150	-5	20	100	-5
8	120	-4.5	21	75	-3.5
10	100	-4	23	70	-3
12	120	-5	24	80	-4.5
14	80	-3.5	26	80	-4
15	80	-3	29	75	-4
16	80	-4	30	100	-5

According to explanations in part D for transmission constrained electricity markets, we simulate the system with space of bidding quantities descretized at precision of 1. The evolution of quantities for each player and objective function minimization are demonstrated in Fig. 10. It can be observed that the proposed algorithm is capable of finding NE for this complex problem. DIWO parameters are also presented in Table XII.

V. CONCLUSION

In this paper we modified DIWO for optimization in descretized spaces and employed it for Nash equilibrium search in games with local NE traps. Performance of DIWO for function optimization was tested through a set of popular test functions in stochastic optimization. Moreover, DIWO was used to minimize objective functions which encode NEs as their minima. A static nonlinear game with multiple local NEs and Cournot model of IEEE30 bus test system with transmission constraints were two experiments studied in this paper. Results showed that DIWO has a good performance for the purpose of global optimization and NE search for games with discrete strategy spaces.

For future work, we are to study the proposed algorithm for NE search in mixed strategy games with numerous equilibria. In addition, evaluating approach of characterizing NEs as minima of a function for games with continuous spaces is our current focus of research.



(a) (b) (c)

Fig. 10. NE search for IEEE30 bus test system with DIWO
a) Strategies evolution for player 1, 2, 3.
b) Strategies evolution for player 4, 5, 6.
c) Objective function minimization

TABLE XII. DIWO PARAMETERS FOR NE SEARCH IN IEEE30 BUS TEST SYSTEM

Symbol	value	Symbol	value
N_0	20	n	3
$iter_{max}$	100	σ_{init}	15
P_{max}	20	σ_{final}	1
S_{max}	3	R	20
S_{min}	1	-	-

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