NASH EQUILIBRIUM SEARCH FOR NONLINEAR GAMES USING EVOLUTIONARY ALGORITHMS

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ABSTRACT

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Finding Nash Equilibrium (NE) for nonlinear games is a challenging work due to existence of local Nash Equilibrium traps. So, devising algorithms that are capable of escaping from trapping in local optima and finding global solutions is needed for analysis of nonlinear games. Evolutionary Algorithms as the popular stochastic global search algorithms can be exploited for this purpose. In this thesis, Nash Equilibrium search approaches for nonlinear games are studied through a number of numerical examples and practical problems.

Coevolutionary programming, evolutionary iterative Nash Equilibrium search, and minimizing objective functions with embedded Nash Equilibria, using evolutionary algorithms, are the main methods discussed in this work. Also, local optimization algorithms are employed in some of the problems for comparison.

For practical simulations we apply the proposed algorithms to several nonlinear games in electricity market models with two to six players. Transmission-constrained electricity markets with linear and nonlinear demand functions and unconstrained electricity markets with a nonlinear total demand are the main case studies in this work. We adopt Invasive Weed Optimization for all the evolutionary computing purposes, and the efficiency of our proposed Coevolutionary Invasive Weed Optimization (CIWO) for finding global NE is shown in the simulations. Likewise, successful results of the proposed Discrete Invasive Weed optimization (DIWO) in NE search for games with discrete strategy spaces are provided.

CHAPTER I

INTRODUCTION

John Nash's formulation of noncooperative game theory was one of the great breakthroughs in the history of social science. Nash Equilibrium (NE) is a solution of a game involving two or more players in which no player has incentive to unilaterally change her action, so any change in strategies by any of the players lead that player to earn less.

In November, 1949, the proceedings of National Academy of Science received a short note from Nash which was published the next year [35]. In this assay, Nash gave general definition of equilibrium for normal-form games, and he neatly sketched an argument using the Kakutani fixed-point theorem to prove that equilibria in randomized strategies must exist for any finite normal-form game. Also, in 1951, he presented his outstanding article *"Noncooperative Games"* [36] in which he argued that his noncooperative equilibrium concept, together with von Neumann's normal form gives a complete general methodology to analyze all games. Furthermore, he showed the efficiency and importance of his proposed equilibrium in a number of interesting examples, illustrating problems which have concerned game theorists ever since, including a game with one Pareto-inefficient equilibria like Prisoners' Dilemma.

Historically, the concept of NE was developed before Nash in literature by a number of famous scientists. Antoine Augostin Cournot in his brilliant book [37], constructed a theory of oligopolistic firms that includes monopolists and perfect competitors as limiting extremes (1838). In fact, we may speak of Cournot as the founder of oligopoly theory [38].
Another prominent work was done by John von Neumann and Oskar Morgenstern who introduce the concept of mixed strategy NE for special case of zero-sum games [39].
Also, Bertrand (1883) to Felner (1949) found specific models of oligopoly which had some applied predictions [40], [41].

Since its development, NE plays an important role in game theory and has been used for modeling problems in a variety of areas like economics, biology, engineering, political science, computer science, philosophy, etc.

In this thesis we aim to find Nash Equilibrium for nonlinear games. Although rigorous mathematical frameworks have been devised to approach games with linear manipulation, however, considerable amount of attention have been dedicated in recent years for NE search in the case of nonlinear games. In this respect, the terms of evolutionary game theory [34] and Natural selection in biology and life science have been employed by economics and engineers to have more insights to the concept of Nash Equilibrium. In fact, evolutionary game theory in a computational scheme is the application of natured-inspired models of change in generations of populations to game theory.

In this study, Evolutionary Algorithms as the popular means of global search are exploited to find NE for nonlinear noncooperative games. We try to put all the common methods of NE search in evolutionary frameworks and analyze their performance through a large number of numerical simulations for finding global NE. The oligopolistic games in electricity markets are the practical problems which are studied in this work.

The remainder of thesis is arranged as follows. Chapter II explains methods and approaches for NE search in games with two or more players. A numerical example is solved with each proposed method to identify the efficiency, advantages and drawbacks of the algorithms and also to have a comparison between evolutionary and non-evolutionary frameworks.

Chapter III provides a large number of practical simulations for different models of games in electricity markets. Moreover, performance of the proposed methods in chapter II for solving complex games with fairly large number of players is investigated.

Finally, chapter IV delivers the decisive message of my work and clears the perspective for future works.

CHAPTER II

METHODOLOGY

Many techniques have been developed for searching Nash Equilibrium (NE) in game theory problems. All the approaches are inspired by NE definition which is maximizing the payoff, given other players' strategies. The simplest method which can be applied to two or three player games, is finding the intersection of best response curves (reaction curves) by drawing or Algebra. For graphical approach, some geometric techniques have been also proposed to come up with more than two player problems [17]. Algebra can improve the method to solve games with several players, but it can be applied to problems with simple mathematical manipulations. This algorithm is commonly used in Cournot or Bertrand models of electricity markets with linear demand functions, using the first-order condition for maximizing each player's payoff [1], [18], [19].

Iterative NE search in which players repeatedly maximize their payoff by turn is another method that is applied to more complex problems. The profit maximization problem which is embedded in this method can be solved by local or global optimization algorithms. In literature, local search is more popular and have been employed in [3], [28] and [26], however in [9], a GA-based algorithm is also presented for profit maximization.

In recent years, with development of Soft Computing [23], and increasing growth of Biomimicry [24], and Bioinspired Computing in a variety of applications, there has been a considerable attention to evolutionary game theory and computational intelligence for game learning and simulation of electricity markets [3], [7]-[9], [13], [14], [22], [32], [33]. Coevolutionary programming is the most popular technique for this purpose. In [3], a novel Hybrid Coevolutionary is applied to solve constrained-transmission electricity markets, and in [8], a GA-based coevolutionary algorithm is exploited to simulate a simple electricity pool. Besides coevolutionary algorithms, learning methods in agent-based approach have also been used to study imperfect competition in electricity markets [5], [20], [21]. In fact, these days, agent-based economics is a rigorous opponent of game theory to simulate electricity markets.

Another approach for searching NE is characterization of NEs in terms of minima of a function and then minimizing this objective function. This method was firstly employed in finding mixed strategy NEs [13], [14], but recently a similar technique was introduced in [7] to identify pure NE in games with a large number of players. It seems that more in investigations are needed to understand the efficiency of this model.

Section 1, provides a quick review for the concept of Nash Equilibrium. In section 2, Coevolutionary Programming to find NE is explained, while Iterative NE Search algorithms are described in section 3. Finally, section 4 summarizes methods of modeling Nash Equilibrium as a minimum of a function. Note that for each section a numerical example is simulated and the results are interpreted.

1. Nash Equilibrium

A general multi-player game consists of an index set $N = \{1, 2, 3, \dots, \mathrm{N}\}$ called player's set and an index set $K = \{1, 2, 3, \dots, \mathrm{K}\}$ as the stages of the game, showing the allowable number of moves for each player. In each stage, players take strategies from a set of strategy spaces $U = \{U_k^i\}$, and receive a payoff of $\pi_i(u^i, u^{-i})$, where $u^i \in U^i$ is the pure strategy for player i, given pure strategy set of others $u^{-i} = \{u^1, \dots, u^{i-1} u^{i+1}, \dots, u^N\} \in U^{-i}$. Pure strategy Nash Equilibrium (NE) is a point where no player can obtain a higher profit by unilateral movement. The satisfying NE condition for the combined strategy $\{u^{i*}, u^{-i*}\}$ is characterized in (2.1).

$$\forall i, \forall u^i \in U^i, \qquad \pi_i\left(u^{i*}, u^{-i*}\right) \geq \pi_i(u^i, u^{-i*}) \tag{2.1}$$

As we will use the term ***local NE*** in this dissertation, here a definition of that from [3] is also provided.

$$\exists \varepsilon > 0 \; such\ that \; \forall i, \forall u^i \in B^{i,\varepsilon}\left(u^{i*}\right), \qquad \pi_i\left(u^{i*}, u^{-i*}\right) \geq \pi_i(u^i, u^{-i*}) \tag{2.2}$$

Where $B^{i,\varepsilon}\left(\hat{u}^i\right) = \{u^i \epsilon U^i \;\; \left\| u^i - \hat{u}^i \right\| < \epsilon\}$

2. Coevolutionary Programming

In [10], coevlolutioanry algorithm (CEA) is defined as "an evolutionary algorithm that employs a subjective internal measure for fitness assessment." The term ***subjective internal measure*** means that fitness for the individuals are measured based on their interaction with each other and this fitness value influences their evolution in some way. This is a general definition for coevolutionary algorithm which most the coevolutionary computation researchers agree, however there are controversy on some topics like what precisely is the nature of interaction? Should the interacting individual be in different populations? Do they have to treat concurrently? [10] The answer to these questions is beyond the scope of this survey, but in this thesis, we focus on multi-population models in which the fitness for individuals is measured by their interaction with individuals in other populations. In the following two parts we define Cooperative and Competitive Coevolutionary Algorithms and present coevolutionary frameworks to find Nash Equilibrium for game theory problems.

A. Cooperative Coevolutionary Algorithm

In Cooperative CEA, each population represents a piece of a larger problem and the populations evolve their own pieces in interaction with each other to solve the larger problem. A general cooperative coevolutionary framework for is explained in Algorithm 1.

Algorithm 1. General framework for Cooperative CEA

1. For population $p_s \epsilon P$, all populations
   1.1. Initialize population p_s
2. For population $p_s \epsilon P$, all populations
   2.1. Evaluate population p_s with collaborators
3. t:=0
4. do
   4.1. For population $p_s \epsilon P$, all populations
      4.1.1. Evolutionary Process to make the next generation
      4.1.2. Evaluate next generation with collaborators
   4.2. t:= t+1
5. Repeat 4 until terminating criteria is met

For evaluating part, each individual is combined with its collaborators from other populations to form a complete solution and the objective function is evaluated. Terminating criteria can be satisfied by falling short of the acceptable tolerance for changes in strategies or exceeding the maximum number of iterations. In evolutionary process, any evolutionary algorithm (EA) can be exploited, like Genetic Algorithm (GA), Particle Swarm Optimization (PSO), Mimetic Algorithm (MA), Simulated Annealing (SA). We employ Invasive Weed Optimization (IWO), a novel EA proposed by Mehrabian and Lucas [2], for all the evolutionary computation purposes throughout this dissertation (See Appendix A).

To put NE search algorithms in the framework represented by Algorithm 1, each player is represented by a population of strategies. The fitness of each strategy is evaluated by selecting the collaborating players from other populations and payoff calculation for that strategy.
Selecting the collaborators is very important in coevolutionary programming to have the best performance and find the real solutions. In [10], a number of attributes for this purpose are named: sample size, selective pressure and credit assignment. Sample size determines the number of collaborators, while selective pressure is the bias we impose on the selection procedure, and credit assignment deals with the fact how to assign one fitness value to each individual from the results of multiple objective function evaluation.

In our proposed coevolutionary programming, for the purpose of NE finding, we set the sample size for each player to 1, i.e. each player takes one collaborator and for our selective pressure we consider two cases: 1) collaborators are selected at random and 2) the best strategies from the last evaluation are taken as the collaborators.
The former was studied in [3] and [8], while the latter was applied in part of the proposed Hybrid Coevolutionary Algorithm with GA and Hill Climbing in [3] for the goal of NE search. In this section we have a comparison between these two cases to find NE for a numerical example of a nonlinear static game. This benchmark will be studied for evaluation of all the proposed methods in this chapter.

a. Numerical Example

This is a nonlinear static game with *local NE traps* [3], which is also analyzed in [3] and [9], and we can consider it as a good benchmark for nonlinear games. The profit function for this game is characterized in (2.3), and the global best responses and the local best responses for this game are illustrated in Fig. 2.1.

$$\pi_1(x_1, x_2) = 21 + x_1 \sin(\pi\, x_1) + x_1\, x_2 \sin(\pi\, x_2)$$
$$\pi_2(x_1, x_2) = 21 + x_2 \sin(\pi\, x_2) + x_1\, x_2 \sin(\pi\, x_1) \tag{2.3}$$

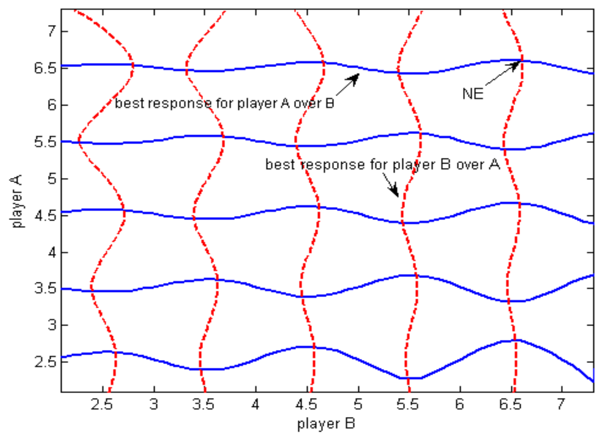
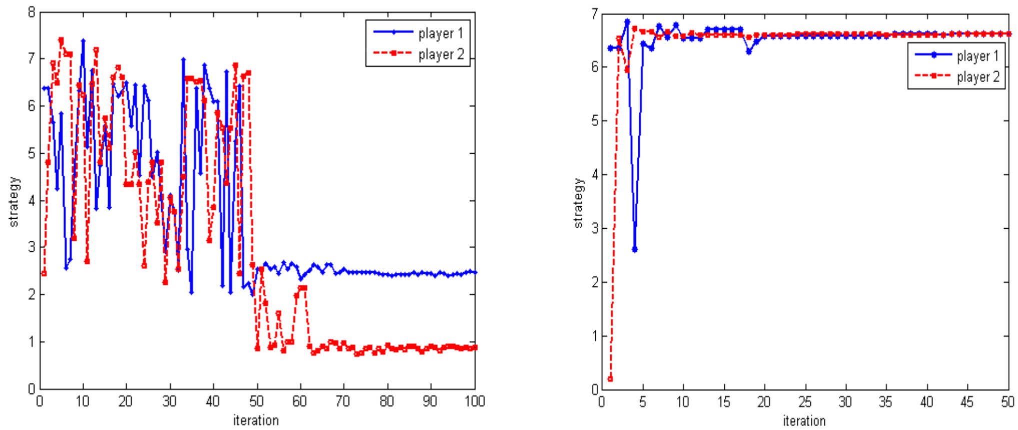


Fig. 2.1. Local and global best responses for the numerical example game

We use IWO for evolutionary process and apply the proposed coevolutionary framework, explained in Algorithm 1 with the both cases described above. The coevolution process for the both cases is present in Fig. 2.2.



(a) (b)

Fig. 2.2. a) Cooperative CIWO with random collaborators
b) Cooperative CIWO with best collaborators.

It is shown that the coevolutionary approach with random collaborators fail to find the global NE while for the second case the strategies quickly converge to the global NE. Note that the both cases have the same number of fitness evaluation and are the same in computational complexity, but the selection pressure we adopt for the second case improves the algorithm. Moreover, for the purpose of comparison with the previously proposed coevolutionary algorithm in [3], we can say that our algorithm is better than simple coevolutionary genetic algorithm in finding the global Nash Equilibrium and also outperforms the hybrid coevolutionary genetic algorithm in number of evaluations and computational complexity.

B. Competitive Coevolutionary Algorithm

In Competitive Coevolution, fitness is evaluated based on direct competition among individuals selected from evolving populations. Three models have been proposed in [11] and [12] for competitive coevolution: 1) Fitness Sharing, 2) Shared Sampling and 3) Hall of Fame. In this study we use Fitness Sharing which is more well-known than the other two methods. In Fitness Sharing, each individual in the population plays with all individuals in other populations (i.e. its payoff is compared with others), and then the individuals in the population is scored according to (2.4).

$$s_i = \sum_{j \in X} \frac{1}{N_j} \tag{2.4}$$

Where X is the set of individuals defeated by individual i, and N_j is the number of individuals in the same population as i that are victorious over individual j. Although Fitness Sharing is the core of our proposed algorithm, but here, we present a modified and novel framework for competitive coevolution to find Nash Equilibrium in game theory problems. This framework is summarized in Algorithm 2.

Algorithm 2. The proposed framework for Competitive CEA to solve games

1. For population $p_s \epsilon P$, all populations
    1.1. Initialize population p_s
2. t:=0
3. do
    3.1. For population $p_s \epsilon P$, all populations
        3.1.1. Calculate Payoffs for population p_s with slected collaborators
        3.1.2. Assign the payoffs for each individual and its collaborators to that individual
        3.1.3. Apply Fitness sharing for population p_s after playing of each individual with all the individuals in other populations
        3.1.4. Evolutionary Process to make the next generation
    3.2. t:= t+1
4. Repeat 3 until terminating criteria is met

Like previous section, each player is represented by a population of strategies, and also the best individuals in the last evaluation are considered as collaborating players, since this selection pressure caused excellent results in the previous section.

a. Numerical Example

Performance of our proposed Competitve CES is assessed by solving the nonlinear game described in section II.2.A.a. We use IWO as the evolutionary algorithm, and the coevolution process to find NE for this problem is illustrated in Fig. 2.3.

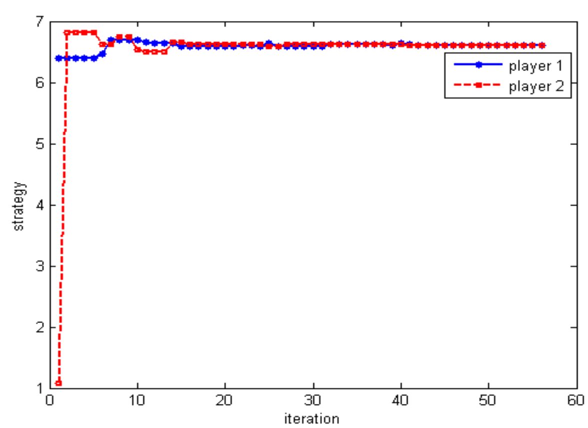


Fig. 2.3. Local and global best responses for the numerical example game

We can see that Competitive Coevolutionary IWO (CIWO) converges to global NE very quickly, and also there are fewer fluctuations compared to Cooperative CIWO in Fig. 2.2.b. However, the computational complexity is more for Competitive approach because of Fitness Sharing process.

3. Iterative NE Search

In Iterative NE Search, players perform the profit maximization by turn until no player changes its strategy. The performance of this method is dependent on the search algorithm adopted. The general framework for this process is summarized in Algorithm 3.

Algorithm 3. Framework for Iterative NE Search

1. Initialize each player's strategy
2. For each player
   2.1. Fix other player's strategies and solve profit maximization problem
3. Repeat 2 until terminating criteria is met

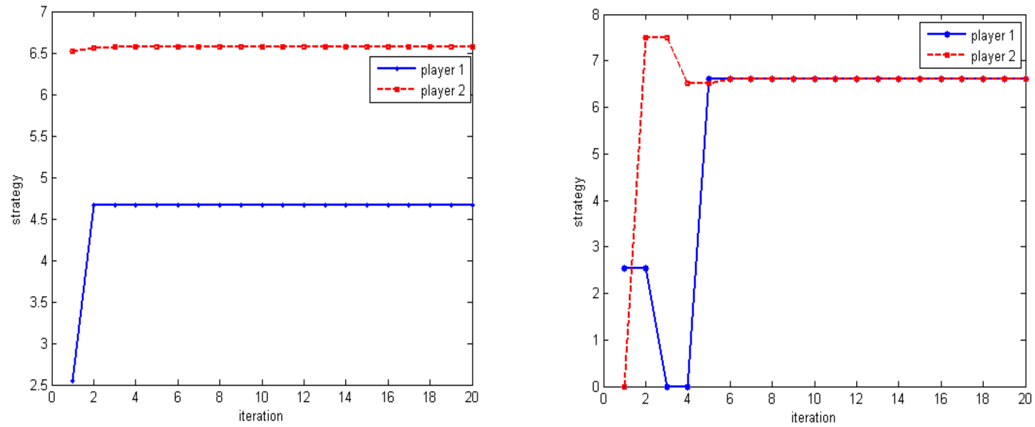
Profit maximization problem for each player can be solved by local or global optimization algorithms. Using local optimization to find NE for games has been studied in [3], [26], [28], while exploiting global optimization was proposed in [9]. In the following parts, both two approaches are described.

A. Local Iterative NE Search

In this method local optimization algorithms are used to solve profit maximization problem. In [28], a penalty interior point algorithm is recruited, and in [26] modified Newton step is used. In all of the simulations in this dissertation, a local optimization software *"fmincon"* in the MATLAB Optimization Toolbox is used, the same as the work done in [3].

a. Numerical Example

We study the performance of this algorithm in solving the numerical example introduced in section II.2.A.a. Two cases are considered for simulation: 1) fixed initial points in all the iterations for the local optimization algorithm, and 2) adaptive initial points. In the latter, the initial points are set to the solution in the last iteration.
The results of simulation for the both cases are featured in Fig. 2.4. It is shown that Local Iterative NE Search with fixed initial points fails to find the global Ne and get stuck at local NE traps. However, the second case is capable of finding NE after a few iterations. Although, this simulation shows the superiority of adaptive approach over the other one but, it is needed to have more investigations to make a general conclusion.



(a) (b)

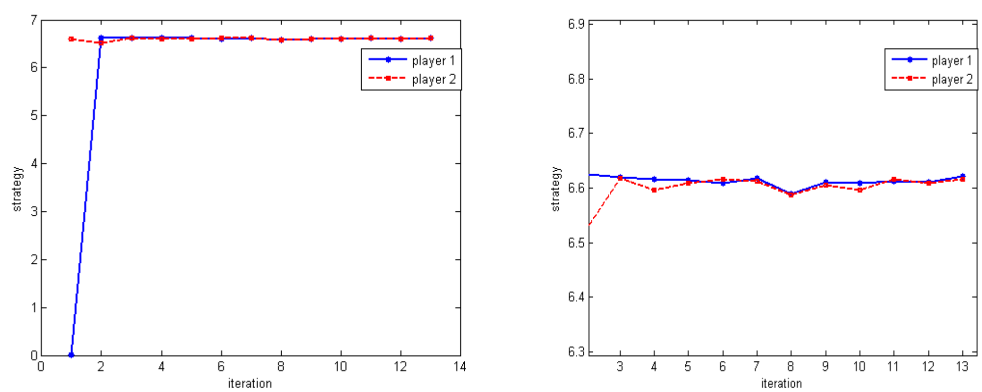
Fig. 2.4. a) Local Iterative NE Search with fixed initial points
b) Local Iterative NE Search with fixed initial points

B. Global (Evolutionary) Iterative NE Search

This method was proposed in [9], using Genetic Algorithm (GA) for the purpose of global optimization. In this part we use Invasive Weed Optimization (IWO) as the evolutionary algorithm and study the performance of this method.

a. Numerical Example

We solve the nonlinear static game explained in section II.2.A.a, using Evolutionary iterative NE Search with IWO. The evolutionary process to find NE for this problem is depicted in Fig. 2.5.a. We can see the algorithm converges to the global NE very quickly.



(a) (b)

Fig. 2.5. a) Local Iterative NE Search with fixed initial points
b) Local Iterative NE Search with fixed initial points

In spite of the capability of the method to find NEs, there are some problems with this algorithm. Firstly, using global optimization in each iteration and for each player takes a long time in the whole process and makes the algorithm inefficient for simulation of games with a large number of players. Furthermore, if the parameters for the evolutionary algorithm aren't appropriately set, then in the case of immature convergence, there will be considerable fluctuations in the solutions even at the final steps of iterative search. This problem is illustrated in Fig. 2.5.b.

4. Nash Equilibrium as a Minimum of a Function

The idea of characterization of Nash equilibria in terms of minima of a function was developed in [7], although a similar approach was previously used in [13] and [14] for identifying mixed NEs in games. In this method, an objective function is defined in which the minima are the NEs, and then any optimization algorithm can be exploited to solve this minimization problem. However, this objective function is driven in an indirect process that makes hard for the local optimization algorithms to find the minima, so a stochastic optimization algorithm should be used.

One of the advantages of constructing an objective function is that we have a factor to assess the efficiency of the calculated solution, by its fitness value in the objective function. The other advantage is that we can apply conventional techniques like deflection, stretching, repulsion, etc. in optimization for the computation of all NEs [13]-[15].

The objective function for each combined strategy u in the strategy space $U$and payoff function J is defined as follows:

$$D(u) = \sum_{i=1}^{N}[max_{u_i' \in U_i} J_i(u_1, \dots, u_{i-1}, u_i', u_{i+1}, \dots, u_N) - J_i(u)] \quad (2.5)$$

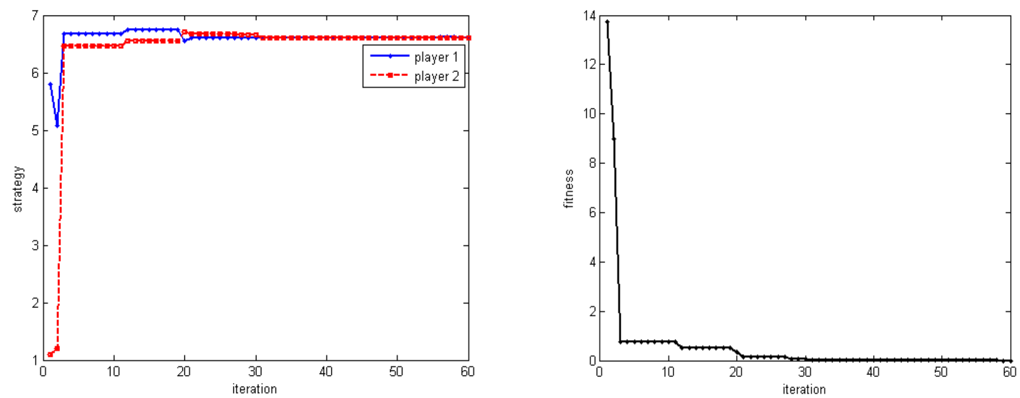
Form the classical definition of Nash Equilibrium, it is easily concluded that the function D is strictly positive, if the combined strategy u is not an equilibrium and equal to zero otherwise, so the NEs are the minima of this function.
As it is evident in (2.5), a maximization problem is embedded in this function, for which direct exhaustive search, local or global optimization can be employed. In games with discrete and not too large strategy spaces, maximization can be performed by sorting the payoffs, but for continuous games, local or global maximization might be useful.

A. Discrete Minimization

In this case, the strategy spaces might be discrete by their nature or can be discretized to small grids with arbitrary precision. So, the objective function is easily calculated for each strategy by exhaustive search and then an stochastic optimization is employed to minimize this function.

a. Numerical Example

We evaluate the performance of this method with our proposed discrete optimization algorithm, Discrete Invasive Weed Optimization (DIWO) for the nonlinear static game presented in section II.2.A.a. A detailed explanation of DIWO is provided in Appendix B. For this problem the strategy spaces are discretized with precision of 0.1. The simulation results to find NE for this problem is depicted in Fig. 2.6. Fig. 2.6.a shows the strategies evolution while Fig. 2.6.b, presents trace of fitness values for the objective function through the evolutionary process. We can see that the algorithm is capable of identifying the global NE with our determined precision.



(a) (b)

Fig. 2.6. a) Strategies evolution with DIWO
b) Objective function minimization with DIWO

B. Continuous Minimization

In this case the strategy spaces are continuous and we don't want to descretize the problem, so we are facing with continuous minimization for objective function and continuous maximization for profit maximization. For minimization any stochastic optimization algorithm can be adopted, and for profit maximization, the same as the approaches presented in Iterative NE Search, we can apply local or global optimization algorithms. The two following parts explain the procedure.

a. Local Profit Maximization

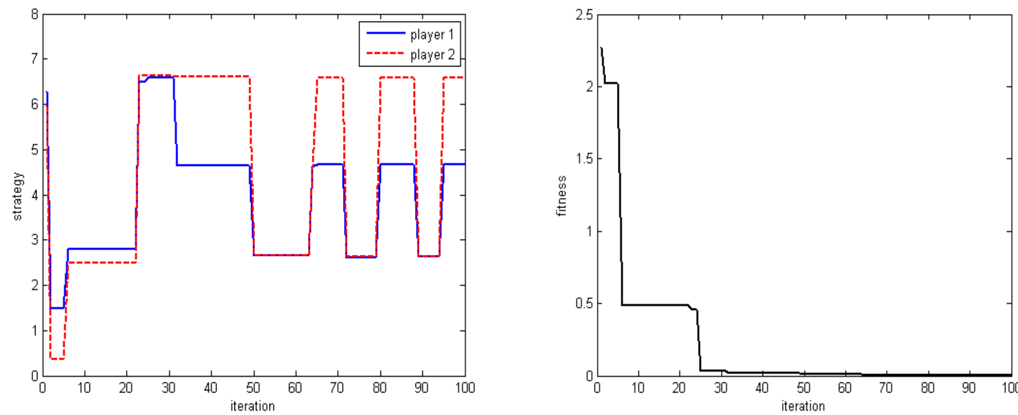
In this approach a local optimization algorithm is exploited to maximize the payoff. In this survey, we use local optimization software "fmincon" in MATLAB Optimization Toolbox for the purpose of profit maximization. Then IWO is employed for objective function minimization.

b. Global (Evolutionary) Profit Maximization

In this approach, an evolutionary optimization algorithm is exploited to maximize the payoff. In this study, for the both cases of profit maximization and objective function minimization, we employ IWO.

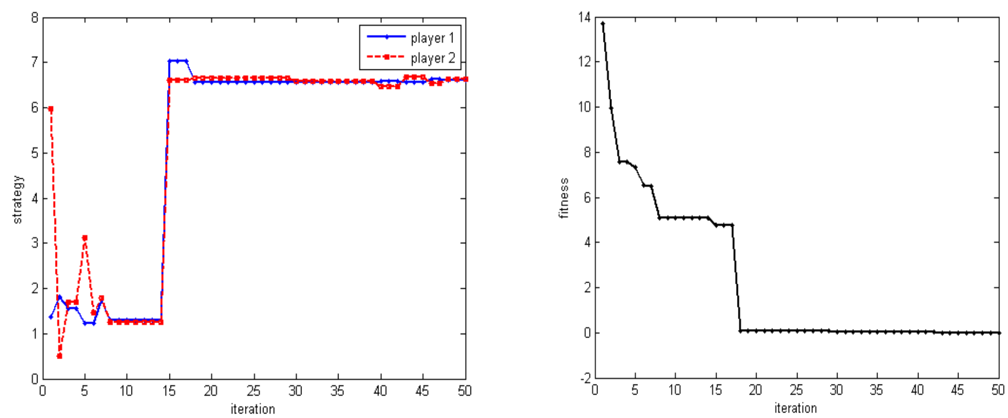
c. Numerical Example

We investigate the efficiency of the two above-mentioned algorithm to find NE for the nonlinear game introduced in section 2.A.a. Fig. 2.7 shows the evolutionary process of IWO for continuous minimization with local profit maximization, and Fig. 2.8 demonstrate the evolutionary process of IWO for continuous minimization with evolutionary profit maximization. We can see that the algorithm with local profit maximization is trapped at local NE, while the evolutionary approach converges to global NE.



(a) (b)

Fig. 2.7. a) Strategies evolution with IWO for continuous minimization with local profit maximization
b) Objective function minimization with IWO for continuous minimization with local profit maximization



(a) (b)

Fig. 2.8. a) Strategies evolution with IWO for continuous minimization with evolutionary profit maximization
b) Objective function minimization with IWO for continuous minimization with evolutioanry profit maximization

It is useful to say that our simulations show that the both methods esp. evolutionary maximization impose high computational complexity and take more time than the other methods introduced in this chapter. However, the advantages mentioned in first part of section II.4 are provided in these methods, and they might be applicable for some special purposes.

CHAPTER III

SIMULATION RESULTS

In this chapter, a number of nonlinear games with two to six players are studied and the effectiveness of the proposed methods in the previous chapter is examined to find NE for these problems.
The principal models that are analyzed in this chapter are transmission-constrained electricity market with linear demand functions, unconstrained electricity market with a total nonlinear demand function, and transmission-constrained electricity market with nonlinear demand functions.

Although transmission-constrained electricity markets with linear demand functions have linear demand curves, but the transmission constraints can cause individual profit functions to have local optima [26]. Actually, reaction curves in this model are discontinuous piecewise linear functions that might make *local NE traps* [3] or even disrupt existence of pure strategy equilibrium for the game [3], [4], [29]. Besides the fact that transmission-constrained electricity market model is a good mathematical example with a complex game structure and local optima, it is an important model for market power analysis in the restructured electricity industry [29]-[30].

In unconstrained electricity markets with total nonlinear demand function, the complexity of the game is due to nonlinearity of demand curves and so local approach in NE search might fail to find global NE. As there is one demand in this model, a uniform price is existed in this market.

Transmission-constrained electricity markets with nonlinear demand functions have the both above-mentioned complexities, and so, analysis and interpretation of this market is very challenging. In this market, if the constraints occur, locational price differences are produced.

This chapter is organized as follows: section 1 is dedicated to transmission-constrained electricity markets with linear demand functions. In section 2, unconstrained electricity markets with a total nonlinear demand are studied, and also a comparison between the proposed methods in chapter 2 is provided in this section. Section 3, consists of problems in transmission-constrained electricity markets with nonlinear demand functions, and finally, three other nonlinear games involving electricity spot market, electricity pool market model and a dynamic nonlinear game are discussed in section 4.

1. Transmission Constrained Electricity Markets with Linear Demand Functions

As it was mentioned in the introduction, transmission constrained electricity market is a good example of complex game for our purpose of *Soft Computing*. Shortly, trading in electricity markets can be represented by the maximization of total welfare subject to the constraints on the system (3.1).

$$\max\left(\sum_j Benefit_j - \sum_i Cost_i \right) \tag{3.1}$$

$$S.T. \begin{cases} Transmission\ thermal\ limits \\ Total\ supply = total\ demand \\ Kirchoff's\ laws \end{cases}$$

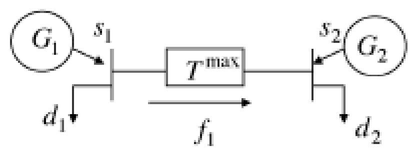
When transmission constraints are binding in the imperfectly competitive market, cournot behavior will produce locational price differences similar to a competitive market with constraints present. This increases the difficulty of computing the profit maximizing condition of the strategic players. The profit maximizing function of each strategic player has an embedded transmission-constrained welfare maximization problem within its major problem. The generation and transmission line constraints are included in the welfare maximization subproblem. The profit function maximization of each utility is given in (3.2).

$$\max\left\{P_i\, q_i - Cost_i \;\middle|\max \sum_j Benefit_j \quad , Transmission\ Constraints\middle|\right\} \tag{3.2}$$

Locational prices (P_i), are determined by the Lagrange multipliers of the locational energy balance equality condition for Kirchoff's laws in the welfare maximization problem which is also the market-clearing problem, here [16], [31].

A. Two-Bus Transmission Constrained Cournot Model.

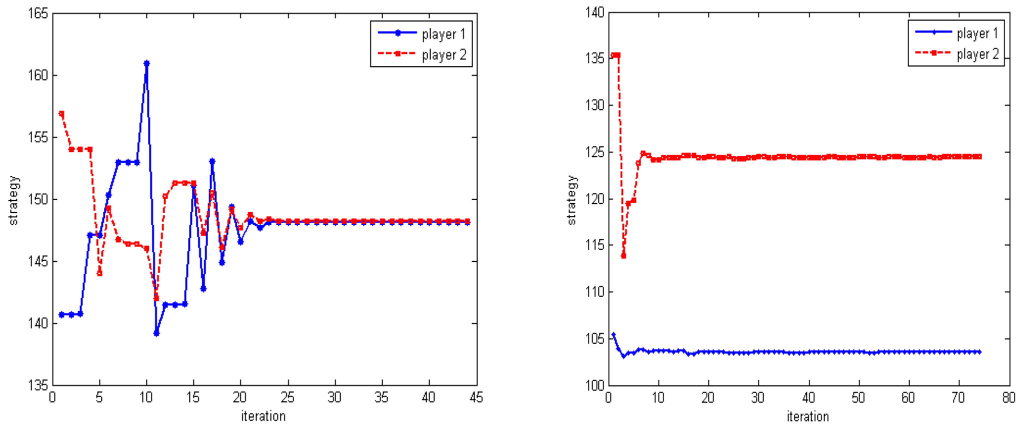
The model, we study in this part is a model with a Generator and a Load at each bus and transmission limit T_{max}, which was introduced in [3] as a qualitatively similar model to California model in [29]. This system is depicted in Fig. 3.1.



$$B_1(d) = -.08d_1^2 + 50d_1 \quad B_2(d) = -.04d_2^2 + 30d_2$$
$$C_1(s) = .01s_1^2 + 10s_1 \quad C_2(s) = .01s_2^2 + 10s_2$$

Fig. 3.1. Two-bus Cournot model

Two cases are considered for the transmission constraints: 1) T_{max} = 80 and 2) T_{max} = 30. The former was solved in [3], using hybrid coevolutionary programming, but the NE obtained for this limit is the same as NE for unconstrained model which is a linear problem to solve. The latter is the one when transmission constraints make sense. We adopted Cooperative CIWO to find NE for the both cases. The coevolution processes for these two problems are illustrated in Fig. 3.2, and also the simulation results for the case of $T_{max} = 30$ are provided in Table 3.1.



(b) (b)

Fig. 3.2. a) cooperative CIWO for two bus model with 80-MW limit. b) cooperative CIWO for two bus model with 30-MW limit.

Fig. 3.2 shows that our proposed algorithm is able to come up with the nonlinearity of transmission constrained market problems in a few steps of coevolution.

Table 3. 1. Cournot Solution for two bus model with 30-MW limit.

s_1	s_2	d_1	d_2	T (1 to 2)	$price_1$	$price_2$
103.53	124.42	133.53	94.42	-30	28.64	22.45

B. Three-Bus Transmission Constrained Cournot Model (Two Generator and Three Load)

In this part, a three node problem which was studied in [5] and [27] is presented. The main purpose is to show the performance of our proposed algorithms when there is no pure NE. The system is presented in Fig. 3.3.

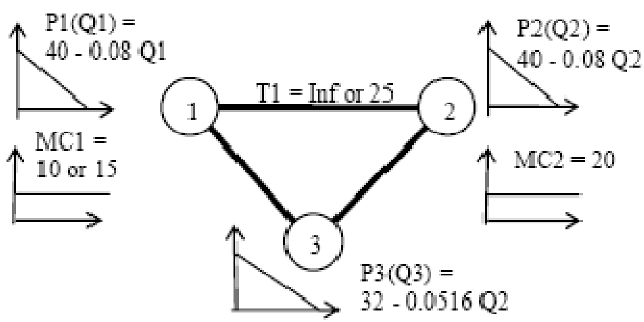
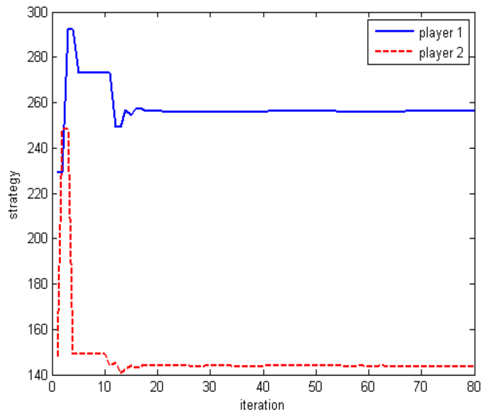
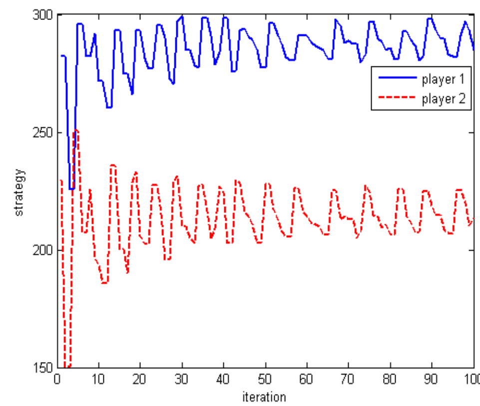


Fig. 3.3 Three-bus cournot model. Generators are in node 1 and node 2.

When MC1=10, there exists one pure NE at $s_1 = 256$ and $s_2 = 144$, however in the case of MC1=15 there is no pure NE. We are to know whether our coevolutionary algorithm is capable to show nonexistence of pure NE or not. In [3], a coevolutionary algorithm was posed which doesn't converge when there is no pure equilibrium. Although, this feature was declared as an advantage of the proposed algorithm, in [5], one of the issues for comparison between Agent-Based approach and Game Theory was their behavior in absence of NE. In Agent-Based approach, with each run agents converge to a plausible equilibrium and by averaging the results a fairly acceptable equilibrium is taken.
The coevolution process for the both cases (MC1=10 and MC1=15), using Coooprative CIWO with the same parameters is depicted in Fig. 3.4. It is shown that the proposed coevolutionary algorithm doesn't converge when no NE exists (MC1=15). But, in Fig. 3.5, it is illustrated that when σ_{final} in CIWO is set to an adequate small value, CIWO converges to an equilibrium. In fact, our proposed coevolutionary algorithm can fulfill the both sides by appropriate tuning of the parameters.



(a)



(b)

Fig. 3.4. a) cooperative CIWO with MC1=10 and $\sigma_{final} = 5$ b) cooperative CIWO with MC1=15 and $\sigma_{final} = 5$

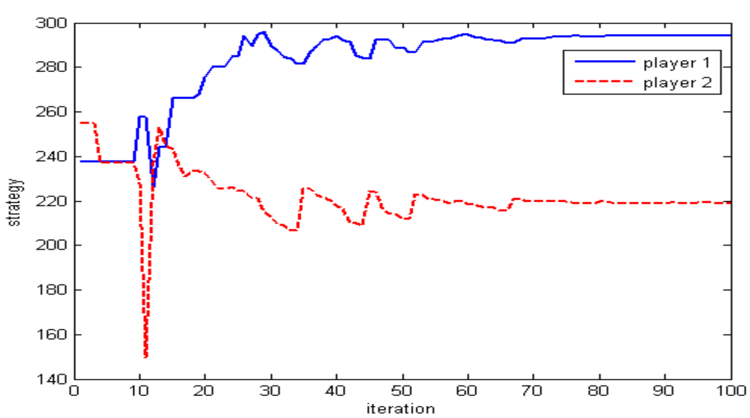
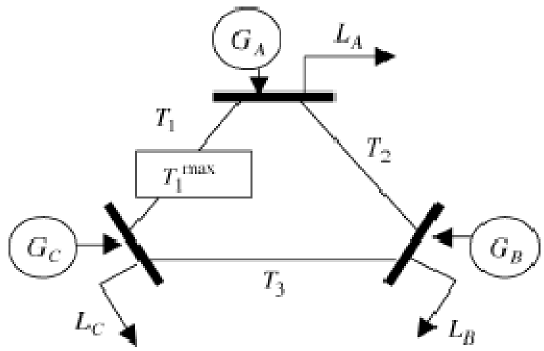


Fig. 3.5. cooperative CIWO with MC1=15 and $\sigma_{final} = 0.1$

C. Three-Bus Transmission Constrained Cournot Model

This is a more complex model with a pure NE at $s_1 = 1106$, $s_1 = 1046$, $s_1 = 995$ which is solved in [3] and [4] with hybrid coevolutionary programming and graphic representation, respectively. This three-bus network is depicted in Fig. 3.6.



$$B_1(d) = -.0555d_1^2 + 108.4096d_1, \qquad C_1(s) = 0.00786s_1^2 + 1.3606s_1,$$
$$B_2(d) = -.0669d_2^2 + 103.8238d_2, \qquad C_2(s) = .010526s_2^2 - 2.07807s_2,$$
$$B_3(d) = -.0637d_3^2 + 105.6709d_3 \qquad C_3(s) = .006478s_3^2 + 8.105354s_3$$

Fig. 3.6. Three-bus cournot model

Here, we use our proposed Cooperative CIWO to find NE in the case of $T_1^{max} = 100$. The coevolution process for Cooperative CIWO is shown in Fig. 3.7. Despite poor performance of the proposed genetic coevolutionary algorithm in [3], Fig. 3.7 shows that our coevolutionary algorithm converges to the optimal solution after a limited number of iterations.

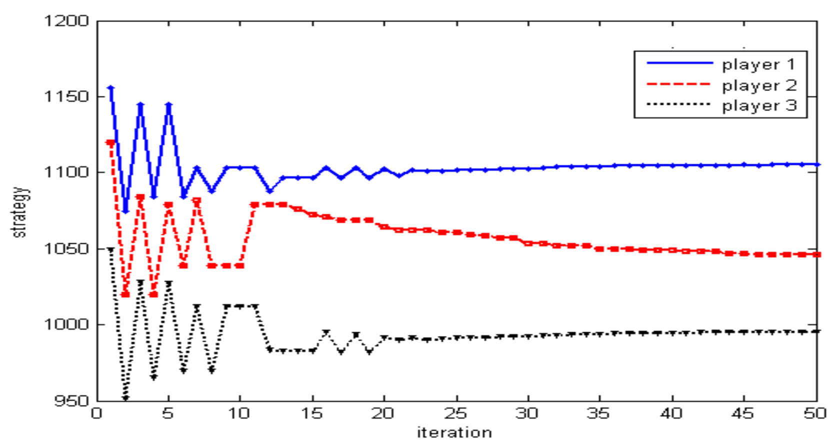


Fig. 3.7. cooperative CIWO for the three-bus model with $T_1^{max} = 100$

D. Four-Bus Transmission Constrained Cournot Model

The model which is studied in this part is a modified oligopoly simulation of a restructured ERCOT market, introduced in [6]. Perfectly competitive, transmission unconstrained and transmission constrained cournot model of this network for three market players and four demand entities were studied in [6]. Here, we solve the transmission constrained cournot model with four market players and four demanders in the case of peak period and constraint of 1000 MW imposed on line 4. The ERCOT equivalent system is featured in Fig. 3.8, and also the demand and cost data for peak condition is provided in Table 3.2.

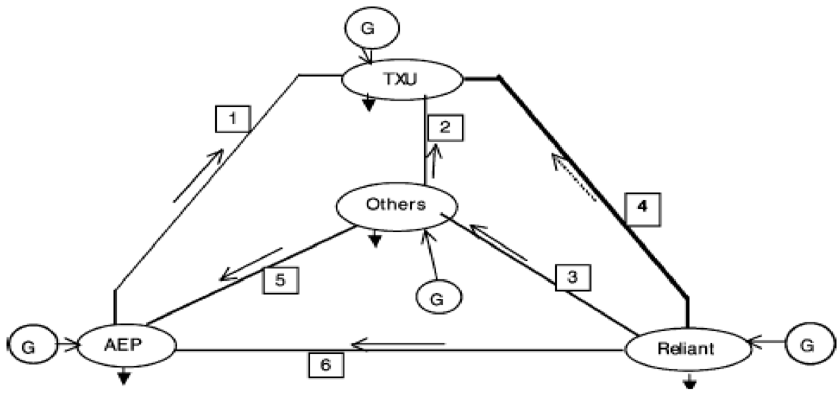


Fig. 3.8. ERCOT equivalent system

Table 3. 2. Data for ERCOT System in Peak Period

		TXU	Reliant & CPSB	AEP	Others
Total Cost	ϕ	0.002255	0.00212	0.00573	0.00478
$C = \frac{1}{2}\phi q^2 + \gamma q$	γ	-11.346	-8.751	3.641	-7.226
Inverse Demand	v	437.4316	528.3013	418.7048	397.099
$P_i = v - oq$	o	0.016399	0.021585	0.05865	0.036837

The coevolution process for this simulation with Cooperative CIWO is depicted in Fig. 3.9. Also the solutions are summarized in Table 3.3.

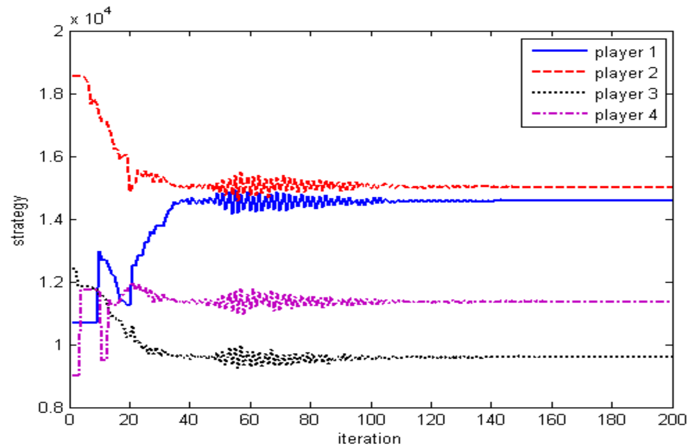


Fig. 3.9. cooperative CIWO for the four-bus model

Table 3. 3. Solutions for ERCOT Model

	TXU	Reliant & CPSB	AEP	Others
Supply (MW)	14585	15007	9609	11366
Demand (MW)	19227	18818	5057	7465

2. Unconstrained Electricity Markets with a Total Nonlinear Demand Function

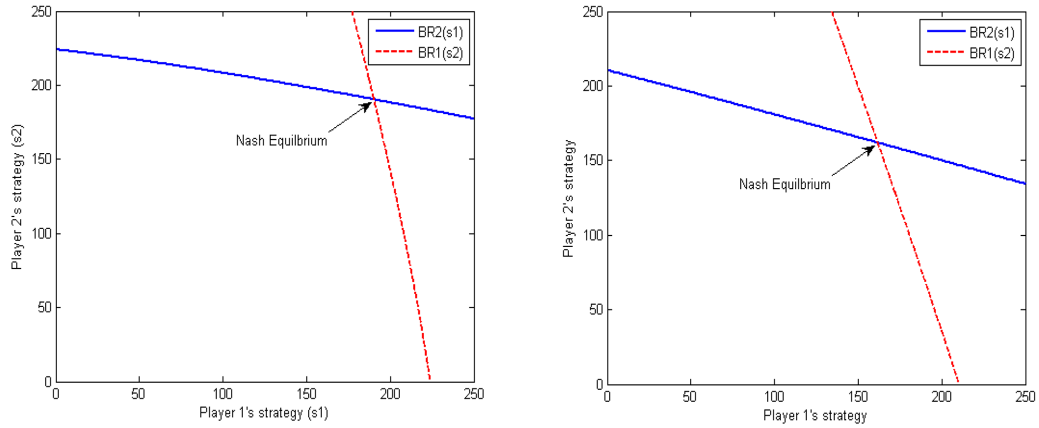
A. Cournot Model with Two Firms

In this section, a problem with two firms and one nonlinear demand curve for the whole system is presented. Two cases with two different nonlinear demand curves are studied: 1) a homographic demand function and 2) an exponential demand curve.
The inverse demand functions for these two cases are provided in (3.3) and (3.4), respectively. The cost functions for the both cases are the same as the cost function for the two-bus constrained model in section III.1.A.

$$P = \frac{13610}{Q+347.5} \tag{3.3}$$

$$P = 37.99 \exp(-.002028\, Q) \tag{3.4}$$

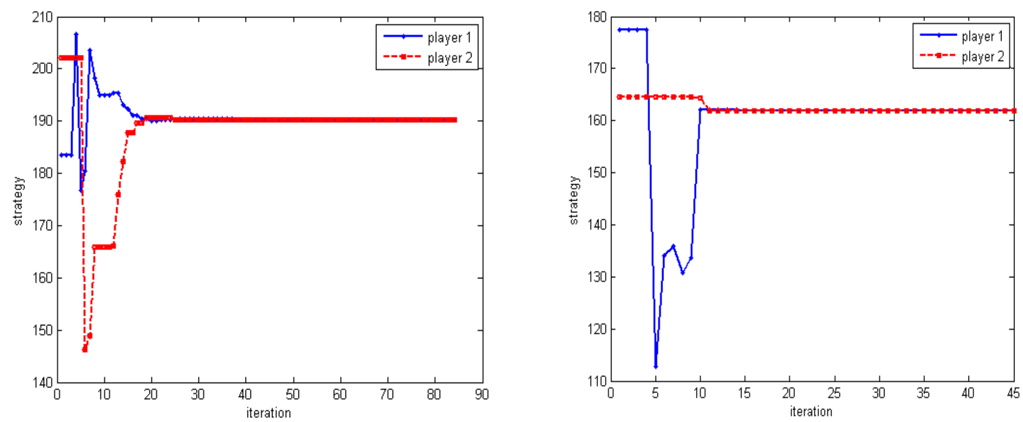
Q is total demand for the market, and P is the clearing price of market.
The global best response functions of these two cases are depicted in Fig. 3.10. As it is shown in the figure, NE for homographic demand function is located at [190, 190], while for the exponential one there is a pure NE at [162,162].



(a) (b)

Fig. 3.10. a) best response functions in two-firm model with homographic demand function
b) best response functions in two-firm model with exponential demand function

The coevolution process for Cooperative CIWO to solve these nonlinear problems are shown in Fig. 3.11. We can see that the proposed coevolutionary algorithm is capable of finding NE for the both model.



(a) (b)

Fig. 3.11. a) Cooperative CIWO coevolution for two-firm model with homographic demand function
b) Cooperative CIWO coevolution for two-firm model with exponential demand function

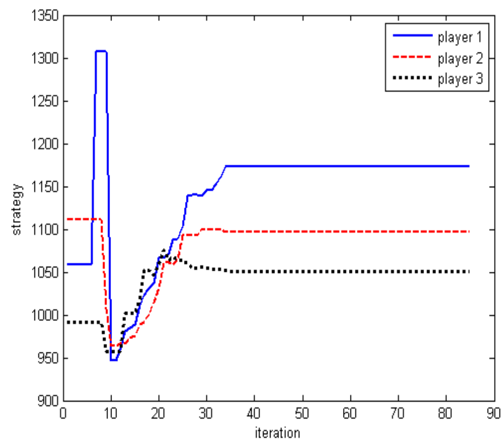
B. Cournot Model with Three Firms

For a more complex problem, in this section we examine the performance of CIWO to find NE for a oligopoly electricity market problem with three firms (players) and one total demand. Like the previous section, we consider two different types of demand curves. The inverse demand functions for these two cases are listed in (3.5) and (3.6). Cost functions for these two models are the same as those in section III.1.C.

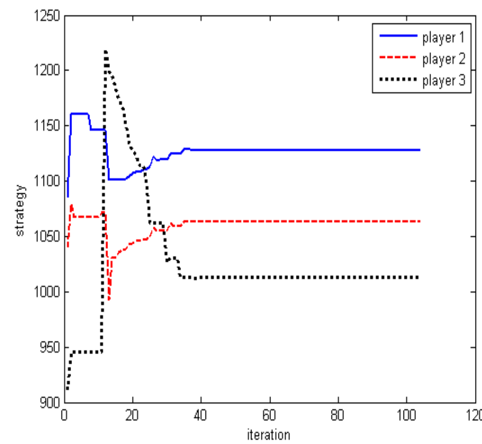
$$P = 2.366 * 10^6 \, Q^{-1.361} \quad (3.5)$$

$$P = 176.3 \exp(-0.0004631 \, Q) \quad (3.6)$$

The coevolution process for these two problems, using Cooperative CIWO is depicted in Fig. 3.12. It is shown that there is a pure NE for the first problem at [1173 1097 1051], and there is one at [1128 1063 1013] for the second problem.



(a)



(b)

Fig. 3.12. a) Cooperative CIWO coevolution for three-firm model for the inverse demand curve in (3)
b) Cooperative CIWO coevolution for three-firm model with exponential demand function in (4)

C. Cournot Model with Four Firms

In this section a cournot model with four firms (players) and one total nonlinear function is studied. Like previous sections, two different types of demand curves are considered. The inverse demand function for these two models are shown in (3.7) and (3.8). The former is a power function and the latter is an exponential one. Cost functions are also the same as those described in section III.1.D. (ERCOT simulation).

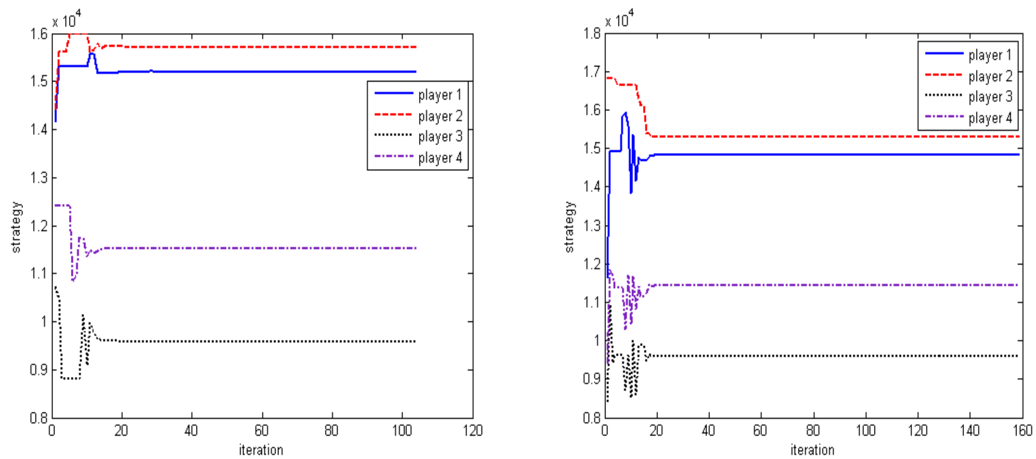
$$P = 1.658 * 10^{14} \, Q^{-2.581} \quad (3.7)$$

$$P = 1692 \exp(-5.219 * 10^{-5} \, Q) \quad (3.8)$$

The solutions for these two systems are provided in Table 3.4. The coevoltion process for the both models with Cooperative CIWO in finding NEs is illustrated in Fig. 3.13.

Table 3. 4. Solutions for Cournot Model with Four Firms

Players	Firm 1	Firm 2	Firm 3	Firm 4
Supply for Power demand function (MW)	15198	15713	9578	11514
Supply for Exponential demand function (MW)	14835	15294	9590	11422



(a) (b)

Fig. 3.13. a) Cooperative CIWO coevolution for four-firm model for the inverse demand curve in (5)
b) Cooperative CIWO coevolution for four-firm model with exponential demand function in (6)

D. Cournot Model with Six Firms

In this section, we study a problem with six firms (players) and one total nonlinear demand function. Although, somehow a large number of players exist in this nonlinear profit maximization problem and so the complexity of the problem is more than the previous ones, this problem can be easily solved by analytic calculation. Actually, the main purpose of our survey in this problem is to investigate the performance of the proposed methods in chapter II for solving games with large number of players and also their ability to come up with *local NE traps* [3].

The demand function and Marginal Costs (MC) for this game are listed in Table 3.5.

Table 3. 5. Data for Six-Firm Cournot Model

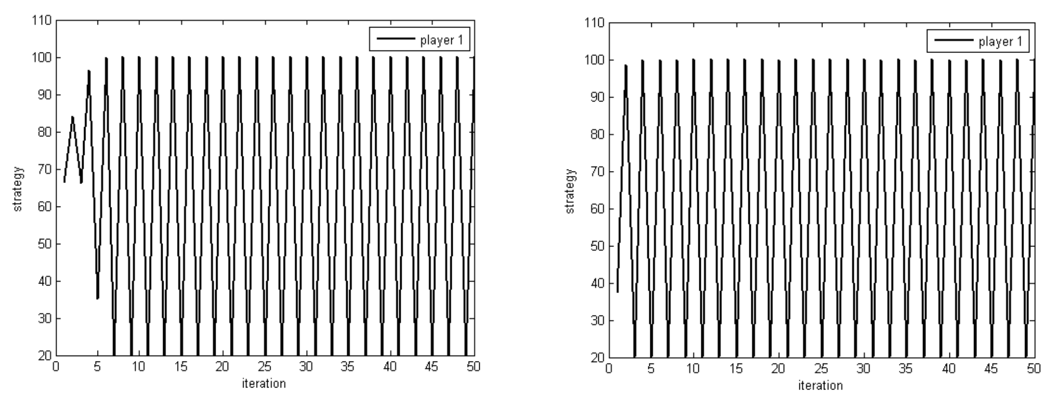
Demand function	MC1	MC2	MC3	MC4	MC5	MC6
$Q = \dfrac{8000}{P}$	20	24	25	22	26	23

The analytic solution to this problem is given as follows. Considering the profit function $\pi_i = q_i . P(Q) - MC_i . q_i$, the first order condition is given by (3.9).

$$P = MC_i - P'(Q) . q_i \tag{3.9}$$

For this problem $P'(Q) = -\frac{8000}{Q^2}$ or $P'(D(P)) = -\frac{P^2}{8000}$ and so the first order condition is $P = MC_i + \frac{P^2}{8000} q_i$ or $q_i = 8000(P - MC_i)/P^2$. Let $MC = \frac{1}{n}\sum MC_i$, where $n = 6$ in this problem, then the aggregate supply function will be $S(P) = 8000\, n\,(P - MC)/P^2$. Equating supply and demand function the equilibrium price will be $P = n.MC/n - 1$. But there is another local equilibrium at $P = \infty$ or $Q = 0$. i.e. in the case of constrained profit maximization problem, we can say that there is a local NE trap in the lower bounds of q_is. In our simulations for all the methods, lower bounds are set to 20.

So, from the previous paragraph, it is concluded that the global Nash Equilibrium for this game is located at [81.6138 40.7843 29.9913 61.1742 21.1945 51.0040] with clearing market price of 28. The simulation results with different methods, explained in chapter II are depicted in Fig. 3.14 and Fig. 3.15.
It is shown that all the algorithms in Fig. 13 are capable of finding Nash equilibrium, however, Local Iterative NE Search and Evolutionary Iterative NE Search which are illustrated in Fig. 3.14 failed to find the equilibrium. It can be seen that for the both approaches strategies fluctuates between the lower bounds and a big value. For the case of comparison between the methods in Fig. 3.15, we can say that Cooperative CIWO, Competitive CIWO, and Discrete Minimization with DIWO converge very quickly, but Continuous Minimization includes an exhaustive process with a large number of fitness evaluations that imposes more time and increases computation complexity. Shortly, the efficiency of the proposed methods in this simulation is ranked as follows: 1) Cooperative CIWO 2) Competitive CIWO 3) Discrete Minimization with DIWO 4) Continuous Minimization with local profit maximization 5) Continuous Minimization with evolutionary profit maximization 6) Iterative Local and Evolutionary NE Search.



(a) (b)

Fig. 3.14. a) Iterative Local Search to solve six-firm cournot model
b) Iterative Evolutionary Search to solve six-firm cournot model

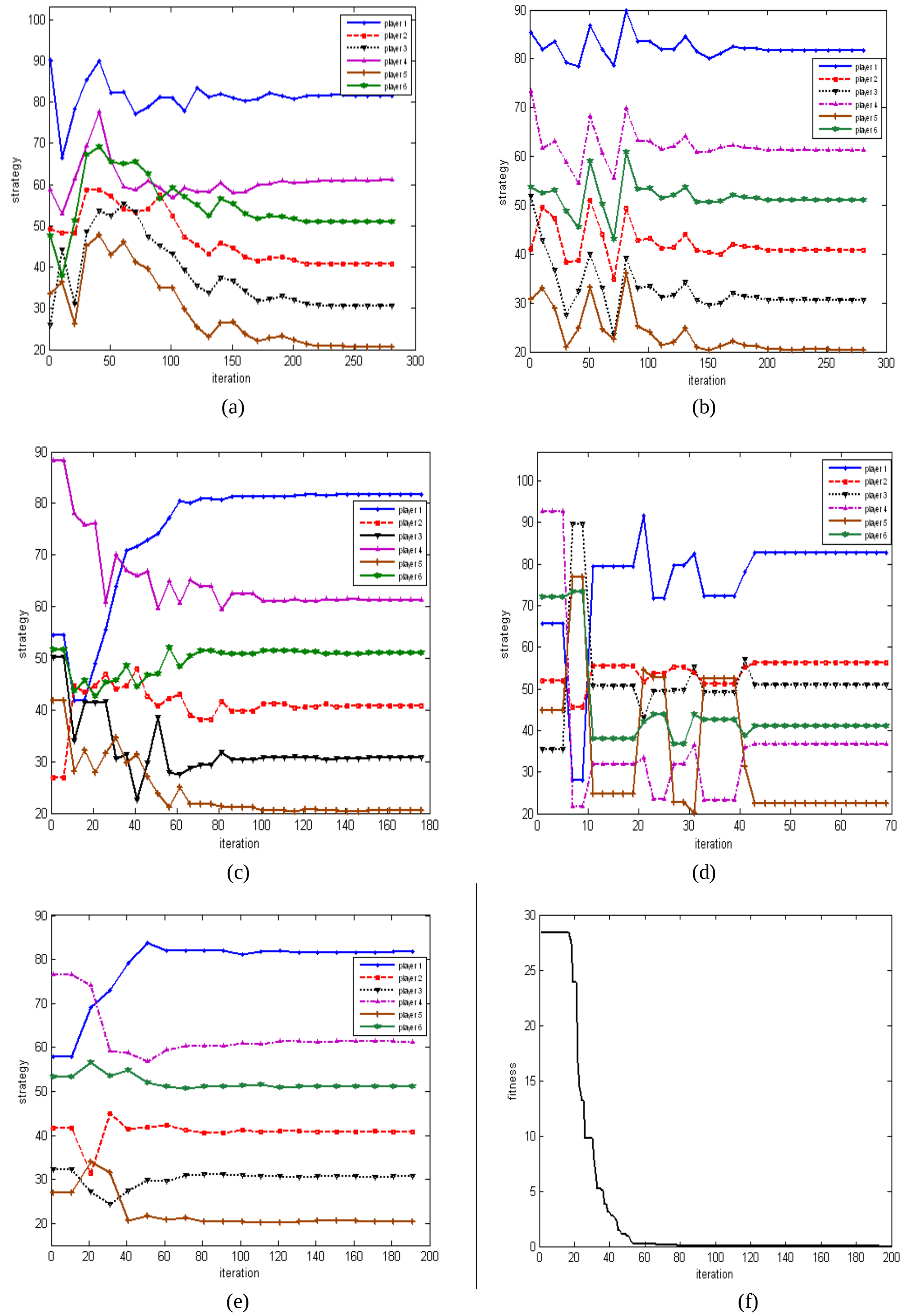


Fig. 3.15. Performance of different methods to solve the six-firm cournol model

a) Cooperative CIWO

b) Competitive CIWO

c) Continuous Minimization with local profit maximization

d) Continuous Minimization with evolutionary profit maximization

e) Discrete minimization with DIWO

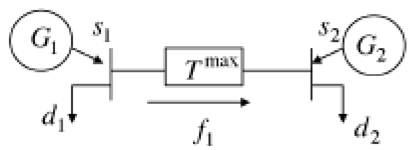
f) DIWO minimization process

3. Transmission Constrained Electricity Markets with Nonlinear Demand Functions

In this section, we study transmission constrained electricity markets. The utility (benefit) functions are given for each demander and the existence of NE is investigated, using two level optimization process described before. In all the examples in this section, utility functions are in the form of power functions which have the required properties of general utility functions like positive slope and convexity.

A. Two-Bus Cournot Model

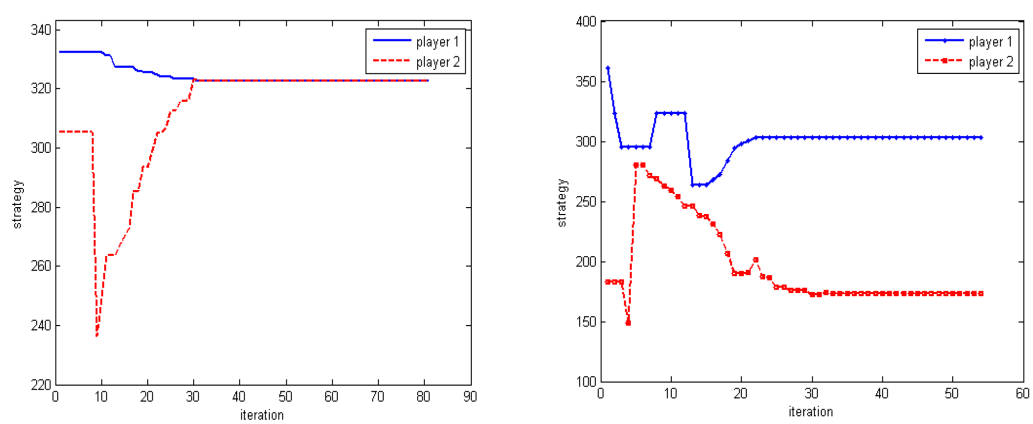
The model for this network is depicted in Fig. 3.16. Two cases are studied for this game with T^{max} = 300 and T^{max} = 30. In the first case, there is a NE at [323 323] with a uniform price across the market, while in the second case, NE is located at [303 173], and locational price differences occur.



$$B_1(d) = 143.8\, d_1^{0.7268} \qquad B_2(d) = 70.41\, d_2^{0.7795}$$
$$C_1(s) = .01 s_1^2 + 10 s_1 \qquad C_2(s) = .01 s_2^2 + 10 s_2$$

Fig. 3.16. Two-bus Cournot model with nonlinear demand

The coevolution process to solve these two problems, using Cooperative CIWO is illustrated in Fig. 3.17, and also the results for T^{max} = 30 are summarized in Table 3.6.



(a) (b)

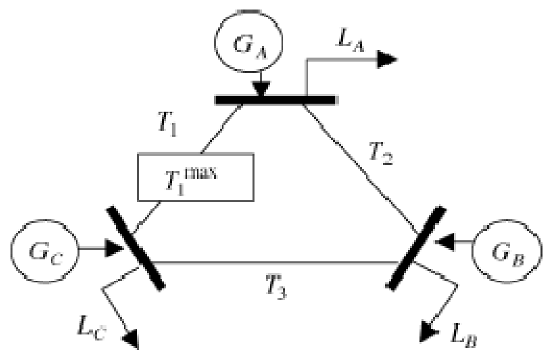
Fig. 3.17. a) Cooperative CIWO for two-bus nonlinear model with T^{max} = 300
b) Cooperative CIWO for two-bus nonlinear model with T^{max} = 30

Table 3. 6. Simulation Results for Two-Bus Nonlinear Model with $T^{max} = 30$

s'_1	s_2	d'_1	d_2	T	p'_1	p_2
303	173	333	143	-30	21	18

B. Three-Bus Cournot Model

For a more complex problem we introduce a three-bus model with nonlinear demand curves and transmission constraint $T_1^{max} = 3000$. The system is presented in Fig. 3.18.



$$B_1(d) = 842.8\, d_1^{0.6576} \qquad C_1(s) = 0.00786 s_1^2 + 1.3606 s_1,$$
$$B_2(d) = 1767\, d_2^{0.5292} \qquad C_2(s) = .010526 s_2^2 - 2.07807 s_2,$$
$$B_3(d) = 1384\, d_3^{0.5719} \qquad C_3(s) = .006478 s_3^2 + 8.105354 s_3$$

Fig. 3.18. Three-bus cournot model nonlinear demand

This game has a pure NE at [1893 1612 1795]. Cooperative CIWO was employed to find NE for this problem and the coevolution process for this game is depicted in Fig. 3.19. It is shown that our proposed algorithm converges to the optimal solution after a few epochs.

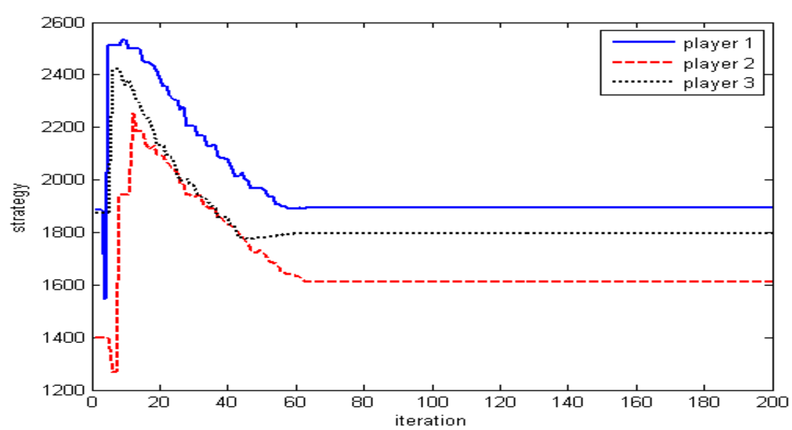


Fig. 3.19. cooperative CIWO for three-bus nonlinear model

4. Some Other Nonlinear Games

A. Uniform-Price Spot Market

In this section, we study the performance of our approach in Discrete Minimization with DIWO on a problem of computation of one Nash Equilibrium for a spot market with two generators. This problem was introduced in [7] and genetic algorithm was employed to solve it. There is a uniform-price electricity market with a price cap of 50 $/MWh and elastic load with inverse demand function $P = -0.083Q_{tot} + 58.33$. Generators bid their full capacity in the form of "price per MWh", and so calculating clearing market price is a nonlinear process which is illustrated in Fig. 3.20. The production capacities and Marginal Costs (MC) for each generator are defined in Table 3.7, accompanied by strategy spaces for U_1 and U_2.

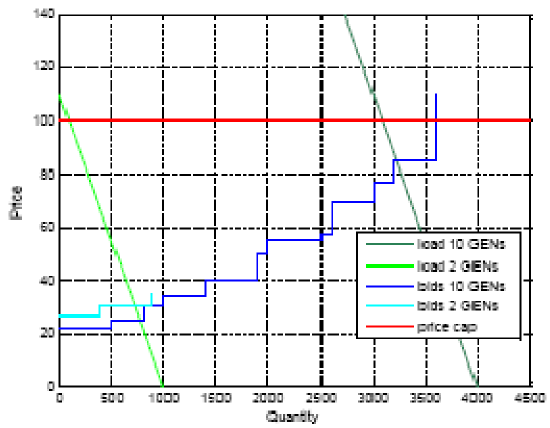
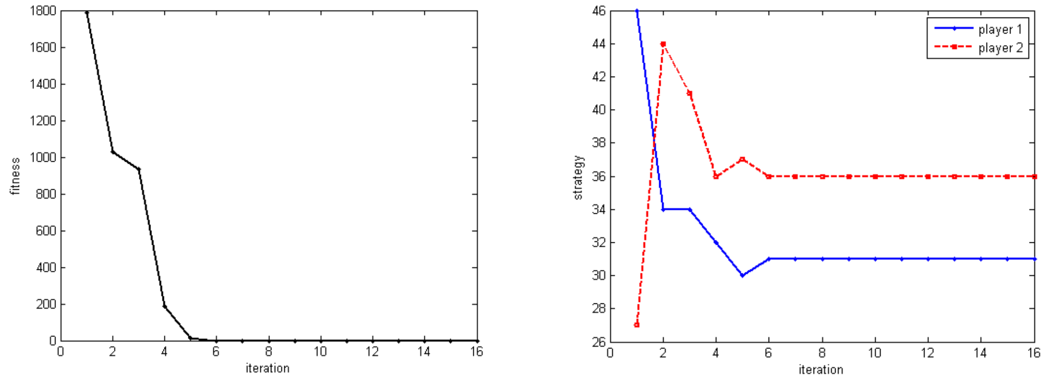


Fig. 3.20. Process of finding clearing market price for some typical elastic load demand curves and supply curves

Table 3.7. Generation Data and Price Strategy Space

	Q_i^{max}	MC_i	U_i
Gen. 1	200	25	{25, 26, … 50}
Gen. 2	300	30	{25, 26, … 50}

Considering the strategy spaces, we are facing a discrete profit optimization problem, and so we use DIWO to find Nash Equilibrium for this game. Trace of fitness values for Discrete Minimization approach described in chapter II is presented in Fig. 3.21.a, and also the strategies evolution is depicted in Fig. 3.21.b. It is shown that after a few iterations the strategies converge to the point [31 36] which is one of the NEs for this problem.



(a) (b)

Fig. 3.21. a) Trace of fitness values for Discrete Minimization approach using DIWO in uniform-price spot market b) strategies evolution for Discrete Minimization approach using DIWO in uniform-price spot market

B. A Simple Electricity Pool

The model studied here is a simple version of that developed by von der Fehr and Harbord (1993) to represent the structure of the UK electricity market. In the UK, power producers bid their generating plant into a pool, where a market price is determined as the bid of the last producer required to satisfy demand. All producers who have bid lower than this receive the market price for their output. This problem was analyzed and solved in [8], using coevolutionary programming with genetic algorithm (GA).
The profit function for this game is characterized in (3.10) and (3.11).

$$\pi_1(p_1, p_2) = \begin{cases} 10\, p_2 & p_2 > p_1 \\ 8\, p_1 & p_1 > p_2 \\ 9\, p_1 & p_1 = p_2 \\ 10\, p_1 & p_2 > 45 > p_1 \\ 0 & p_1 > 45 \end{cases} \tag{3.10}$$

$$\pi_2(p_2, p_1) = \pi_1(p_1, p_2) \tag{3.11}$$

The pure strategy equilibrium for this model can be intuitively deducted when one of the players bids the maximum (in this problem 45) while the other bids less than 36.
The coevolution process for this game using Cooperative CIWO is illustrated in Fig. 3.22. We can see that the strategy for the maximum taker is quickly converge to 45, but the strategy for the other one fluctuates for the values less than 36.

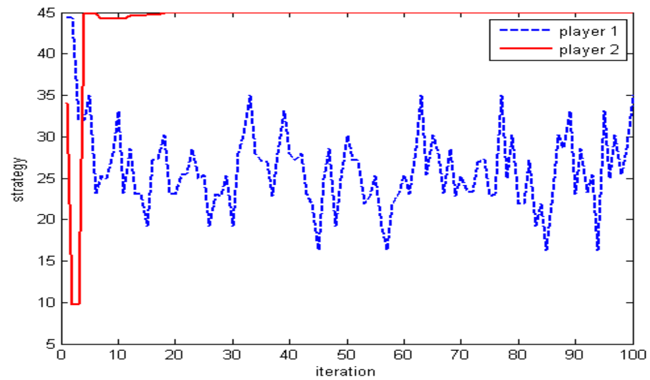


Fig. 3.22. Cooperative Coevolutionary process for the electricity pool model in (8) and (9)

C. A Dynamic Nonlinear Game

In this section, we study the performance of our proposed coevolutionary algorithm to find NE for dynamic nonlinear games. This is a numerical example with two players and two stages which was introduced in [9] and Iterative Evolutionary Search with gentic algorithm was recruited to solve this problem. The payoff for this problem is shown in (3.12).

$$\pi_i(x_{i1}, x_{i2}) = \sum_{t=1}^{2}(1 - y_{1t} - y_{2t})y_{it} - \frac{1}{2}x_{it}^2 \qquad (3.12)$$

$$y_{it} = y_{i,t-1} + x_{it}$$

y_{i0} is given

The coevolution process to find Nash Equilibrium for this problem, using Cooperative CIWO is depicted in Fig. 3.23. We can see that the strategies converge to the global NE after a few iterations.

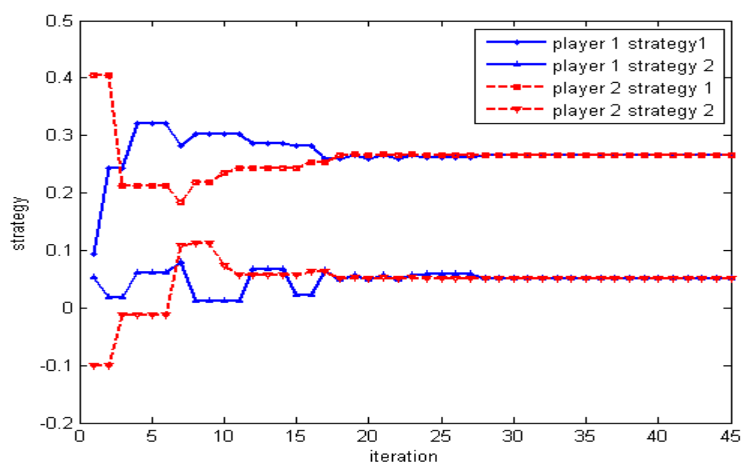


Fig. 3.23. Cooperative Coevolutionary process for the posed dynamic nonlinear game

CHAPTER IV

CONCLUSION

In this thesis, we studied NE search approaches to solve nonlinear games. Coevolutionary programming, Iterative NE search, and NE as a minimum of a function were the major frameworks for our proposed methods. We tried to introduce the best model for each method so we studied the performance of the algorithms with a numerical example and also we provide a comparison between the approaches. Consequently, the proposed methods were applied to a large number of games in electricity market models with two to six players. Transmission-constrained electricity markets with linear and nonlinear demand functions and unconstrained electricity markets with a nonlinear total demand were the main case studies in this work. We adopted Invasive Weed Optimization for all the evolutionary computing purposes, and our proposed Coevolutionary Invasive Weed Optimization (CIWO) was capable of finding global NE for all the problems we studied. Also, we showed its novel bilateral ability when there is no pure NE. Likewise, the proposed Discrete Invasive Weed optimization (DIWO) showed successful results in NE search for games with discrete strategy spaces.

For future works, we can study the performance of the proposed algorithms in finding all the Nash Equilibria in a game. Also, the methods can be employed to identify mixed strategy NE in linear and nonlinear games. Furthermore, due to recent emergence of agent-based economics and its distinctive behavior toward uncertainty in the markets and irrationality of the players, and also its ability for selecting the most plausible equilibrium among several equilibria or managing nonexistence of NE, it is suggested to use above-mentioned evolutionary algorithms as learning mediums for games simulation in an agent-based approach.

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APPENDICES

Appendix A. An Introduction to Invasive Weed Optimization (IWO)

Invasive weed optimization was developed by Mehrabian and Lucas, in 2006. IWO algorithm is a bio-inspired numerical optimization algorithm that simply simulates natural behavior of weeds in colonizing and finding suitable place for growth and reproduction. Despite its recent development, it has shown successful results in a number of practical applications like optimization and tuning of a robust controller, optimal positioning of piezoelectric actuators, developing a recommender system, antenna configuration, etc. To model and simulate colonizing behavior of weeds for introducing a novel optimization algorithm, some basic properties of the process is considered:

1) "A finite number of seeds are being dispread over the search area (initializing a population);
2) Every seeds grows to a flowering plant and produces seeds depending on their fitness (reproduction);
3) The produced seeds are being randomly dispread over the search area and grow to new plants (spatial dispersal);
4) This process continues until maximum number of plants is reached; now only the plants with higher fitness can survive and produce seeds, others are being eliminated (competitive exclusion). The course continues until maximum iterations is reached and hopefully the plant with best fitness it the closest to the optimal solution."

Some of the distinctive properties of IWO in comparison with other evolutionary algorithms are reproduction, spatial dispersal, and competitive exclusion.

In Invasive Weed Optimization algorithm the process begins with initializing a population. It means that a population of initial solutions is randomly generated over the problem space. Then members of the population produce seeds depending on their relative fitness in the population. In other words, the number of seeds for each member is beginning with the value of s_{min} for the worst member and increases linearly to s_{max} for the best member. For the third step, these seeds are randomly distributed over the search space by normally distributed random numbers with mean equal to zero and an adaptive standard deviation. The equation for determining the standard deviation (SD) for each iteration is presented in (A.1).

$$\sigma_{iter} = \frac{(iter_{max} - iter)^n}{(iter_{max})^n}\left(\sigma_{initial} - \sigma_{final}\right) + \sigma_{final} \tag{A.1}$$

where $iter_{max}$ is the maximum number of iterations, σ_{iter} is the SD at the current iteration and n is the nonlinear modulation index. The produced seeds, accompanied by their parents are considered as the potential solutions for the next generation. Finally, a competitive exclusion is conducted in the algorithm. It means that after a number of iterations the population reaches its maximum, and an elimination mechanism should be employed. To this end, the seeds and their parents are ranked together and the ones with better fitness survive and are allowed to reproduce. Pseudocode for IWO algorithm is as follows:

Algorithm A.1. Psuedocode for IWO algorithm

1. Genearte random population of N_0 solutions
2. i=: 1
3. do
   3.1. Compute maximum and minimum fitness in the colony
   3.2. For each individula $w \in N$
      3.2.1. Compute number of seeds of w, corresponding to its fitness
      3.2.2. Randomly distribute generated seeds over the search space with normal distribution around the parent plant (w)
      3.2.3. Add the generated seeds to the solution set, N
   3.3. If $N > p_{max}$
      3.3.1. Sort the population N in descending order of their fitness
      3.3.2. Truncate population of weeds with smaller fitness until $N = p_{max}$
   3.4. i=: i+1
4. Repeat 3 until the maximum number of iterations

The set of parameters for IWO is provided in Table A.1.

Table A.1. IWO Parameters

Symbol	Definition
N_0	Number of initial population
it_{max}	Maximum number of iterations
dim	Problem dimension
p_{max}	Maximum number of plant population
s_{max}	Maximum number of seeds
s_{min}	Minimum number of seeds
n	Nonlinear modulation index
$\sigma_{initial}$	Initial value of standard deviation
σ_{final}	Final value of standard deviation

Appendix B. Discrete Invasive Weed Optimization (DIWO)

Due to IWO's distinctive properties, its global and local abilities for exploration and exploitation, and also its successful results in a considerable number of applications after a short time of its development, we are motivated to introduce Discrete Invasive Weed Optimization (DIWO). DIWO is the modified version of IWO, suitable for discrete optimization problems like vehicle routing, job scheduling, graph coloring, quadratic assignment, routing for telecommunication networks, etc.
The framework for DIWO is the same as IWO's, but some considerations are taken for exploration in discrete search spaces. The psuedocode for DIWO is given as follows:

Algorithm B.1. Psuedocode for DIWO algorithm

5. Genearte random population of N_0 from the set of pheasible solutions
6. i=: 1
7. do
    7.1. Compute maximum and minimum fitness in the colony
    7.2. For each individula $w \in N$
        7.2.1. Compter number of seeds of w, corresponding to its fitness
        7.2.2. Randomly select the seeds from the pheasible solutions around the parent plant (w) in a neighborhood of radius R with normal distribution
        7.2.3. Add the generated seeds to the solution set, N
    7.3. If $N > p_{max}$
        7.3.1. Sort the population N in descending order of their fitness
        7.3.2. Truncate population of weeds with smaller fitness until $N = p_{max}$
    7.4. i=: i+1
8. Repeat 3 until the maximum number of iterations

The process for computing the seeds and also competition exclusion is completely the same as IWO, but seeds generation has been modified to random selection of solutions from the hypercube of radius R in the dim-dimentional space of feasible solutions around the plant with a normal distribution.

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